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Linear Programming Part III

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Industrial production, the flow of resources in the economy, the exertion of military effort in a war theater-all are complexes of numerous interrelated activities. Differences may exist in the goals to be achieved, the particular processes involved, and the magnitude of effort. Nevertheless, it is possible to abstract the underlying essential similarities in the management of these seemingly disparate systems.

George B. Dantzig

III. THE SIMPLEX METHOD

A. Basic Solutions

1. Changing inequalities to equations The feasibility set of a linear programming problem is given by a set of inequalities. It is usually easier to work with equations than with inequalities, and so the first step in the simplex method is to convert the system of inequalities into a system of equations.

This conversion is accomplished by introducing new variables into the problem. We can illustrate this procedure with the Fromage Cheese Company example of Section I. For each feasible mixture (x, y) of the two assortments, we define *slack variables* $u, v,$ and w to represent, respectively, the amount of Cheddar, Swiss and Brie cheeses that will remain in stock after the packages have been prepared. The problem we want to solve then becomes:

Find nonnegative numbers x, y, u, v, w such that

$$M = 4.5x + 4y \text{ is maximized}$$

subject to the constraints :

$$\begin{aligned} 30x + 12y + u &= 6000, \\ 10x + 8y + v &= 2600, \\ 4x + 8y + w &= 2000. \end{aligned} \tag{13}$$

2. Solving systems of equations Before reading this section, you should review the Gauss-Jordan method for solving systems of linear algebraic equations; this method is described in Appendix III.

Consider then the following problem: Given the three equations:

$$\begin{aligned} 8x - 8y + 2u + 4v + 2w &= -14, \\ 4x + 2y - 2u - v + 7w &= 29, \\ 1x + 4y + 3u + 5v + 7w &= 2 \end{aligned} \tag{14}$$

solve for u, v, w in terms of x and y .

This problem can be solved in a very systematic way. For notational convenience, we deal only with the matrix of coefficients:

$$\begin{array}{ccccc} x & y & u & v & w \\ 8 & & & & \end{array} \quad (15)$$

Step 1 Eliminate u from each row (equation) except the first row, where u will be made equal to 1.

This requires two operations:

a) Divide every entry of the first row by the coefficient of u ; this number is called the *pivot element* and it is boxed in (15). This operation yields the matrix:

$$\begin{array}{ccccc} x & y & u & v & w \\ 4 & & & & \end{array} \quad (16)$$

b) Eliminate u from the second row by replacing the second row with the sum of the second row and twice the first row. Similarly, eliminate u from the third row by replacing the third row with the sum of the third row and -3 times the first row. By the "first row," we mean the first row of (16). This yields:

$$\begin{array}{ccccc} x & y & u & v & w \\ 4 & & & & \end{array} \quad (17)$$

Step 2 Eliminate v from all rows of (17) except the second. Again, there are two operations:

a) Divide the second row by the pivot. that is, the coefficient of v , and'

b) Add -2 times new row 2 to row 1 and add new row 2 to row 3. The result of these two operations is:

$$x \quad y \quad u \quad v \quad w \tag{18}$$

Step 3 Eliminate w from all rows except the third. Our operations for this step will be performed on the matrix of (18):

- a) Divide the third row by the pivot 7, the coefficient of w in the third row;
- b) Add 5 times row 3 to row 1 and add -3 times row 3 to row 2. The final matrix is:

$$x \quad y \quad u \quad v \quad w \tag{19}$$

This matrix corresponds to a system of equations equivalent to the original one, but with a simpler structure:

$$\begin{aligned} -9x + 10y + u &= 3, \\ 7x - 8y + v &= -7, \\ -x + 2y + w &= 4, \end{aligned} \tag{20}$$

from which we may read off

$$\begin{aligned} u &= 3 + 9x - 10y, \\ v &= -7 - 7x + 8y \\ w &= 4 + x - 2y \end{aligned} \tag{21}$$

Now there are infinitely many particular solutions to this system. They are produced by assigning arbitrary numerical values to x and y and using Eq. (21) to calculate the corresponding values of u , v , and w . For example, setting $x = 1$ and $y = -2$ gives $u = 32$, $v = -30$, $w = 9$.

One particular solution is singled out. This is the one obtained when x and y are both zero. In this case, we have

$$x = y = 0, \quad u = 3, \quad v = -7, \quad w = 4.$$

This is called a *basic solution* and u , v , w are called *basic variables* while x and y are the *nonbasic variables*.

For the original system of equations, we could have designated any two variables as nonbasic and asked for the other three in terms of these. The solution procedure would have been essentially the same, although the matrices at each stage would not have looked the same. In the final matrix of the problem, a basic variable will be represented with exactly one entry equal to 1 and all other entries in that column equal to 0. Each basic variable has the "1" in a different row from all the other basic variables. The values of the basic variables in the basic solution appear in the right-hand-most column of the matrix.

The Gauss-Jordan procedure described in Appendix III will always yield a matrix from which the basic and nonbasic variables can be identified readily.

3. Basic solutions and the LP problem We have rewritten the linear programming problem about the Fromage Cheese Company in the form: Find nonnegative values of x , y , u , v and w so that $M = 4.5x + 4y$ is maximized and such that the equations (13) are all satisfied. Any solution of the system (13) for which all variables are nonnegative is called a *feasible solution*. Any solution of Eq. (13) for which M is maximized is called an *optimal* solution. The linear programming problem is to find a feasible solution which is also optimal. The Fundamental Existence Theorem of Linear Programming states that if there exists an optimal, feasible solution, then there exists one which is also basic. Stated geometrically, this is just the result that linear functions on polygonal convex sets achieve their extreme values at the vertices.

For the cheese example, the solution

$$x = 0, \quad y = 0, \quad u = 6000, \quad v = 2600, \quad w = 2000.$$

is both feasible and basic. The basic variables are u , v , and w . Geometrically, this solution is located at the vertex where the two edges $x = 0$ and $y = 0$ intersect.

This particular solution gives $M = 0$, which is clearly not optimal. We can increase $M = 4.5x + 4y$ by increasing either x or y . One way to go about this is to concentrate on increasing one of the variables. Since a unit increase in x boosts M more than a unit increase in y , it is reasonable to begin by making x as large as possible, while keeping $y = 0$. When $y = 0$, Eq. (13) can be written

$$\begin{aligned} u &= 6000 - 30x, \\ v &= 2600 - 10x, \\ w &= 2000 - 4x. \end{aligned} \tag{22}$$

From these equations, it is apparent that increases in x have the effect of decreasing u , v , and w . To retain a feasible solution, we must insure that these variables remain nonnegative. From the equation for u in (22), we see that x can be increased to 200 before u becomes negative. Similarly, v does not become negative until x exceeds 260 and w does not become negative until x exceeds 500.

Thus we can increase x up to 200 without driving the other variables below zero, but we cannot increase x beyond 200. If we let $x = 200$, we obtain (using Eq. 22) a new basic feasible solution:

$$x = 200, y = 0, u = 0, v = 600, w = 1600$$

for which the revenue has increased: $M = (4.5)(200) = 900$.

Geometrically, we have moved from the vertex (0,0) to the vertex (200,0) which is the intersection of the two edges $y = 0$, $u = 0$.

The step we have just described is the heart of the simplex method of solving linear programming problems. In the next section, we will describe the simplex method in more detail by carrying out all the steps necessary to solve the Fromage Cheese Company example. We can describe the general outline a little more carefully here, however.

First, we need to start with a basic feasible solution. We begin by adding a slack variable to each constraint. This gives us a total of $n + m$ variables in the problem. The solution in which each of the original n variables is set equal to 0 will be a basic, feasible solution. The basic variables will be the m slack variables. Geometrically, we have located ourselves at the vertex where the edges $x_1 = 0$, $x_2 = 0, \dots, x_n = 0$ intersect. Generally, this will be a highly nonoptimal solution.

Second, we seek a new solution in which one of the x_i becomes basic and one of the slack variables, say u_j , becomes nonbasic. The simplex method advises us on how to choose x_i and u_j so that the new solution remains feasible and increases M . The new solution is obtained from the old one by using the Gauss-Jordan technique. Geometrically, we move from the initial vertex to the vertex where the edges $x_l = 0, \dots, x_{j-1} = 0, u_j = 0, x_{j+1} = 0, \dots, x_n = 0$ intersect. This step is repeated a sufficient number of times until M is made as large as it can be.

B. The Simplex Method

The first step in the simplex algorithm to solve the problem of maximizing $M = \mathbf{c} \cdot \mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ is to replace the system of inequalities by equations. This can be done by the addition of slack variables.

The Fromage Cheese Company problem can be formulated as: Find nonnegative values of X, y, u, v, w such that:

$$\begin{aligned}
 30x + 12y + u &= 6000, \\
 10x + 8y + v &= 2600, \\
 4x + 8y + w &= 2000. \\
 -4.5x - 4y + M &= 0
 \end{aligned}
 \tag{23}$$

and so that M is as large as possible.

We write the matrix of coefficients of Eqs. (23) in a special form, called the *extended simplex tableau* (Tableau 5.1).

Tableau 5.1

x	y	u	v	w	M		
[30]	12	1	0	0	0	6000	u
10	8	0	1	0	0	2600	v
4	8	0	0	1	0	2000	w
-4.5	-4	0	0	0	1	0	M
↑							

The first three rows of the tableau indicate that we have a basic feasible solution with basic variables $u, v,$ and w . (The basic variables at each stage appear again to the right of the last column). The value of these basic variables when the nonbasic ones (x and y at this stage) are zero are indicated in the final column. The entry in the lower right-hand corner of the tableau gives the current value of M .

The bottom row of the tableau represents the equation $-4.5x - 4y + M = 0$. This equation remains valid for all x and y , by the definition of M . The fact that x and y appear with negative coefficients in this row means that if either x or y is increased, the value of M must also be increased for the equation to hold.

We want to increase M , but at the same time retain a basic feasible solution. To do this, we need to select one nonbasic variable and make it basic while we pick a basic variable and change it to a nonbasic one. This process is described more succinctly by saying we pick one variable to *enter the basis* and one to *leave the basis*.

Any variable which appears with a negative coefficient in the bottom row of the tableau can be selected to enter the basis. In practice, the variable whose coefficient has the largest absolute value is chosen, because this gives the largest immediate increase in M . There is no guarantee, however, that this practice will lead to the optimal solution in the smallest number of steps.

Suppose we decide to pick the variable in the j th column to enter the basis. This is often indicated by drawing a short arrow under the coefficient of this variable in the bottom row; we have done this in Tableau 5.1.

When a variable enters the basis, its value is going to increase. This is going to affect the other variables in each of the other equations represented in the tableau. The other nonbasic variables are going to remain equal to 0. Thus the variables which are now basic are going to change to preserve the equations. But each basic variable occurs with nonzero coefficient in exactly one row of the tableau, so it is affected by only one equation.

We really don't care how a currently basic variable will change as long as it remains nonnegative. We lose feasibility, however, if it should become negative. Now a basic variable starts out as positive, so we have only to worry about its decreasing. It will decrease only if the coefficient of the j th variable (the one that is entering the basis) in the row containing the basic variable is *positive*. If that coefficient is zero or negative, the basic variable will not decrease. To illustrate this point, suppose we have the equation

$$2x + 0y - 5z + u = 375$$

and u is a basic variable. If the variable z or the variable y is picked to enter the basis, there is no problem: increasing z or y by any positive amount can only cause an increase of u . But if x is chosen to enter the basis, then x can only be increased to $375/2$ before u becomes negative.

This observation sets the stage for explaining how to select a variable to leave the basis if the j th variable is going to enter the basis. We compute the ratio of the entries in the last column of the tableau to the *positive* entries of the j th column. The row which gives the smallest ratio contains the variable which will leave the basis.

Let's illustrate this for Tableau 5.1. The negative coefficient with greatest absolute value in the bottom row is -4.5 . Thus we will let x enter the basis. The other entries in the x -column are all positive so compute all the ratios of the entries in the last column to the corresponding elements in the first column.

Tableau 5.2

	x	y	u	v	w	M		
$\frac{6000}{30}$	[30]	12	1	0	0	0	6000	u
$\frac{2600}{10}$	10	8	0	1	0	0	2600	v
$\frac{2000}{4}$	4	8	0	0	1	0	2000	w
	-4.5	-4	0	0	0	1	0	M
	↑							

The smallest ratio is 200. It occurs in the first row and the basic variable of that row is u . Thus, u will leave the basis when x enters. The *pivot entry* is the coefficient of the variable which is entering the basis which appears in the row of the lowest ratio.

The critical next step in the simplex algorithm is to use the Gauss-Jordan procedure to make the pivot element 1 and to make all the other entries in that column zero, including the entry in the bottom row of the tableau. This is necessary if the variable of that column is to satisfy the definition of being basic. The Gauss-Jordan procedure is used because it guarantees that the system of equations represented by the new tableau will be equivalent to the original system of equations.

When we apply the Gauss-Jordan procedure to Tableau 5.2, we obtain Tableau 5.3.

Tableau 5.3

	x	y	u	v	w	M		
	1	$2/5$	$1/30$	0	0	0	200	x
	0	4	$-1/3$	1	0	0	600	v
	0	$32/5$	$-2/15$	0	1	0	1200	w
	0	$-11/5$	$3/20$	0	0	1	900	M
		↑						

As before, we can read off a basic feasible solution from the tableau. The basic variables are x , v , w . Their values appear in the final column. The non-basic variables are y and u and their value is 0. The current value of M is 900.

The bottom row of Tableau 5.3 contains a negative number. This tells us that we have not yet reached an optimal solution. There is only one

negative entry and that appears in the y -column. This means that in the next step, the variable y is going to enter the basis. To find the variable that will leave the basis, compute the ratios of the last column to the positive entries in the y - column; see Tableau 5.4.

Tableau 5.4

	x	y	u	v	w	M		
$\frac{200}{2/5}$	1	$2/5$	$1/30$	0	0	0	200	x
$\frac{600}{4}$	0	[4]	$-1/3$	1	0	0	600	v
$\frac{1200}{32/5}$	0	$32/5$	$-2/15$	0	1	0	1200	w
	0	$-11/5$	$3/20$	0	0	1	900	M
		\uparrow						

The smallest ratio (150) occurs in the second row. The basic variable of that row is v . Hence v will enter the basis when y enters. The pivot entry is 4. The Gauss-Jordan procedure is used as before. We obtain a new tableau:

Tableau 5.5

	x	y	u	v	w	M		
$\frac{140}{1/15}$	1	0	$1/15$	$-1/10$	0	0	140	x
	0	1	$-1/12$	$1/4$	0	0	150	y
$\frac{240}{2/5}$	0	0	[2/5]	$-8/3$	1	0	240	w
	0	0	$-1/30$	$11/20$	0	1	1230	M
			\uparrow					

In this feasible solution, $x = 140$, $y = 150$, $w = 240$, $u = v = 0$ and $M = 1230$. This is still not the best we can do because there remains a negative entry in the bottom row. The variable u should be brought back into the basis. To find the variable which should leave, we need to compare the two ratios $140/(1/15) = 2100$ and $240/(2/5) = 600$. We do not compute a ratio for the second row because the coefficient of u in this row is negative. Since $600 < 2100$, the basic variable of the third row-which is w -will leave the basis. The result of applying the Gauss-Jordan procedure with pivot entry $2/5$ is:

Tableau 5.6

x	y	u	v	w	M
-----	-----	-----	-----	-----	-----

1	0	0	1/6	-1/6	0	100	x
0	1	0	-1/12	5/24	0	200	y
0	0	1	-4	5/2	0	600	u
0	0	0	5/12	1/12	1	1250	M

The new basic feasible solution is

$$x = 100, y = 200, u = 600, v = w = 0, \text{ and } M = 1250.$$

Since none of the entries of the bottom row of Tableau 5.6 is negative, it is not possible to increase M any more. We have arrived at a basic, feasible, optimal solution of the problem. We see again that the best mixture is 100 packages of the Fancy assortment and 200 packages of the Deluxe. We also read off from this solution that $v = z = 0$ means that all of the Swiss and Brie cheeses will be used up, but since $u = 600$, there will be 600 ounces of Cheddar remaining.

If you re-examine Fig. 5.4, you will see that the effect of the simplex method applied to the cheese example is to start at the vertex $(0,0)$ of the feasibility set and then to move in a counterclockwise manner around the boundary until the vertex $(100,200)$ is reached.

Note how the iterative step of the simplex method replaces one equality constraint by an inequality constraint and one inequality by an equality. This corresponds geometrically to a move from one vertex of the feasibility set to an adjacent vertex.

Dantzig's simplex method provides a systematic way of moving from one basic feasible solution to another basic feasible solution which increases the value of M . The simplex method is an iterative algorithm; that is, it calls for the repetition of the same sequence of basic steps. We have seen that the iteration stops when all entries in the bottom row of the extended tableau are positive. When this happens, we have found an optimal, feasible solution. The iteration may stop for another reason, however. Suppose some entry in the bottom row of the tableau is negative, so that at the next step the associated variable is to become basic. Now suppose that *every* entry in that column is negative or zero. This means that there is no limit on how much we may increase that variable. No matter how large we make it, the other variables will remain nonnegative and we will retain feasibility. On the other hand, the larger we can make that variable, the larger we can make M . Thus, there is no limit to the size of M . In this case, the linear programming problem has *no* optimal solution. The simplex method not only delivers the solution when the problem has one, it also tells you when no solution exists.

C. The Use of Artificial Variables

We have seen that it may be useful to introduce slack variables into linear programming problems. They help convert inequalities into equations and often provide an initial basic solution. In this section I will show that it is sometimes useful to introduce yet another set of new variables.

Let's start by considering the breakfast problem of Section I.D. We may write this problem as:

Find nonnegative numbers x and y such that

$$\begin{aligned}x + 3y &\geq 3, \\ 38x + 24y &\geq 50,\end{aligned}$$

and for which $M = -4x - 6.5y$ is maximized.

To convert the inequalities to equations, we add slack variables u and v so the problem becomes

Find nonnegative numbers x , y , u , and v so that

$$\begin{aligned}x + 3y - u &= 3, \\ 38x + 24y - v &= 50, \\ 4x + 6.5y + M &= 0,\end{aligned}\tag{24}$$

and M is as large as possible.

The variables u and v measure the amount of iron and protein, respectively, that is in excess of the daily requirement. Rather than "slack" variables, they might more appropriately be called *surplus* variables in this problem.

Because the coefficients of u and v are -1 instead of + 1 in Eq. (24), the system of equations does not give a basic solution. This is not a happy state of affairs, because the simplex method must start off with a basic feasible solution. If we multiply the first two equations of Eq. (24) by -1 we do obtain an equivalent system which is basic:

$$\begin{aligned}-x - 3y + u &= -3, \\ -38x - 24y + v &= -50, \\ -4x + 6.5y + M &= 0,\end{aligned}\tag{25}$$

The basic solution here is $x = y = 0$, $u = -3$, $v = -50$. Unfortunately, this solution is not *feasible*, because u and v are negative. Again, we cannot make use of the simplex method.

One way out of these difficulties lies in the introduction of *artificial variables* for each constraint equation having no basic variable. These are "artificial" variables because they do not represent any real quantity in the process being modeled, in contrast to the slack variables which do correspond to real features.

The new artificial problem has the form: Find nonnegative numbers x, y, u, v, a, b such that

$$x$$

for M as large as possible.

In this statement, p is an unspecified but very large positive number and $M' = M - pa - pb$. It is important to note two things at this point:

1. If a basic feasible solution exists for the original problem, then it will constitute a basic feasible solution for the artificial problem (with $a = b = 0$ being nonbasic variables);
2. The simplex method can be applied to the artificial problem.

The result of applying the simplex method will be an optimal, basic feasible solution of the artificial problem. I claim that in this basic feasible solution, the variables a and b will both be 0. For, if either one were positive, then the quantity $M' = M - pa - pb$ could be increased even more by decreasing a or b . This would contradict the fact that we are at an *optimal* solution of the artificial problem. This verifies the claim. Since $a = b = 0$, the optimal value for M' is the optimal value for M and we have an optimal, basic feasible solution for the original problem.

The extended tableau for the artificial problem Eq. (26) has the form
Tableau 5.7

x	y	u	v	a	b	M'		
1	3	-1	0	1	0	0	3	a
38	24	0	-1	0	1	0	50	b
4	6.5	0	0	0	0	1	0	M'
				p	p			

In this tableau, the variables a and b are not quite basic because of the presence of the p 's in the bottom row. We can make them basic by replacing the last row by the last row minus p times the sum of row 1 and row 2. This yields a tableau (Tableau 5.8) in which a and b are basic.

Tableau 5.8

x	y	u	v	a	b	M'		
1	3	-1	0	1	0	0	3	a
[38]	24	0	-1	0	1	0	50	b
4	6.5	0	0	0	0	1	0	M'
-39 p	-27 p	p	p	0	0	0	-53 p	
↑								

We now apply the simplex algorithm as described in Section III.B. Since p is a large positive number, the negative entry with the largest absolute value in the bottom row is 4 -39 p . Thus in the first step of the simplex method, the variable x will enter the basis. The standard computation of ratios gives $3/1 = 3$ in the first row and $50/38 = 1\ 6/19$ in the second row. Thus b will leave the basis. The pivot element is 38 and the Gauss-Jordan procedure produces a new tableau (Tableau 5.9).

Tableau 5.9

x	y	u	v	a	b	M'		
1	$\frac{45}{19}$	-1	$\frac{1}{38}$	1	$-\frac{1}{38}$	0	$\frac{32}{19}$	a
1	$\frac{12}{19}$	0	$-\frac{1}{38}$	0	$\frac{1}{38}$	0	$\frac{25}{19}$	x
0	$\frac{151}{38}$	0	$\frac{2}{19}$	0	$-\frac{2}{19}$	1	$-\frac{100}{19}$	M'
0	$-\frac{45}{19}p$	p	p	0	$\frac{39}{38}p$	0	$-\frac{32}{19}p$	
↑								

Since there are negative numbers in the bottom row of this tableau, the simplex method continues through another iteration. This time the variable y will enter the basis and a will leave. The pivot element is N. The Gauss-Jordan procedure yields Tableau 5.10.

x	y	u	v	a	b	M'	
1	0	$-\frac{19}{45}$	$\frac{151}{90}$	$\frac{19}{45}$	$-\frac{1}{90}$	0	$\frac{32}{45}$ y
1	0	$\frac{4}{15}$	$-\frac{1}{30}$	$-\frac{4}{15}$	$\frac{1}{30}$	0	$\frac{13}{15}$ x
0	0	$\frac{151}{90}$	$\frac{209}{3420}$	$-\frac{151}{90}$	$-\frac{209}{3240}$	1	$-\frac{364}{45}$ M'
0	0	0	0	p	p	0	

Now none of the entries in the bottom row of the tableau are negative (remember that p is a very large positive number). Hence we have arrived at an optimal, feasible solution for the breakfast problem. We can read off the solution from Tableau 5.10:

$$x = 13/15, \quad y = 32/45, \quad M = -m = 364/45$$

Thus Sherry should mix $13/15$ ounces of Brand X cereal with $32/45$ ounces of Yukkies. This mixture will provide $(13/15) + 3(32/45) = 3$ mg of iron and $3.8(13/15) + 2.4(32/45) = 5$ g of protein so that both minimal nutritional requirements are exactly met. The cost of this mixture is $364/45$ or about 8.1 cents.

D. The Condensed Tableau

In this section we will present a slightly streamlined representation of the simplex method. Instead of working with the extended tableau, we perform the necessary calculations on a smaller tableau. This *condensed* tableau is especially useful for small scale LP problems that are going to be solved by paper-and-pencil calculations. It was introduced by E. M. L. Beale and Steven Vajda in the 1950's and extensively used by Albert W. Tucker. It is sometimes referred to as the *Tucker Tableau*.

We begin by examining more closely the effect on the extended tableau of a single iteration of the simplex method. Suppose we are about to perform the step which puts the variable y into the basis and removes the variable w . We represent the tableau before iteration schematically:

		y		Any other column		w	
Pivot row	...	p	...	q	...	1	...
		.					
		.					
		.					
Any other row	...	r	...	s	...	0	...
		.		.		.	
		.		.		.	
		.		.		.	

In performing the iteration, the first step is to divide each element of the pivot row by the pivot element, which in this case is denoted by p . The effect of this step on the tableau is to change its form to:

		y		Any other column		w	
Pivot row	...	1	...	q/p	...	$1/p$...
		.					
		.					
		.					
Any other row	...	r	...	s	...	0	...
		.		.		.	
		.		.		.	
		.		.		.	

Next, the new pivot row is used to make all other entries in the y -column (or pivot column) equal to 0. If the entry in some row in the pivot column is r , then that row is replaced by that row minus r times the pivot row. Performing this step produces the after-iteration tableau:

		y		Any other column		w
Pivot row	...	1	...	q/p	...	$1/p$...
		.				
		.				
		.				
Any other row	...	0	...	s		

The idea behind the condensed tableau presentation is to eliminate from the extended tableau as many columns as there are basic variables. If a variable u is basic at some step, then the column corresponding to it is omitted. Thus the w column would not appear in the before-iteration tableau of our example.

What happens after the iteration? Since w is no longer basic, it will be represented by a column. Since y has become basic, no y column will appear. A simple way to take care of this is to move the w column into the slot previously occupied by the y column.

We will illustrate the condensed tableau in a particular example. Consider the Fromage Cheese Company example once again. Before the first iteration in which x enters and u leaves the basis, the tableaux are:

Tableau 5.11 Before Iteration

x	y	u	v	w	M	
30	12	1	0	0	0	6000
10	8	0	1	0	0	2600
4	8	0	0	1	0	2000
-	-4	0	0	0	1	0
4.5						

(a) Extended Tableau

x	y		
30	12	6000	u
10	8	2600	v
4	8	2000	w
-4.5	-4	0	M

(b) Condensed tableau

After the first iteration, they look like:

Tableau 5.12 After Iteration

x	y	u	v	w	M
-----	-----	-----	-----	-----	-----

u	Y
-----	-----

1	2/5	1/30	0	0	0	200
0	4	-1/3	1	0	0	600
0	32/5	-	0	1	0	1200
		2/15				
0	-11/5	3/20	0	0	1	900

(a) Extended Tableau

1/30	2/5	200	x
-1/3	4	600	v
-	32/5	1200	w
2/15			
3/20	-11/5	900	M

(b) Condensed tableau

The condensed tableau has the advantage of keeping track of the essential information, while discarding the inessential. In this example, the extended tableau requires the linear programmer, or her computer, to perform computations and keep track of 28 numbers at each step. For the condensed tableau, this is reduced to 12 numbers.

Noting that the entry $s - (rq)/p$ can be written as $s + q(-r/p)$, we can write the transformation rules for changing a pre-iteration condensed tableau to its post-iteration format:

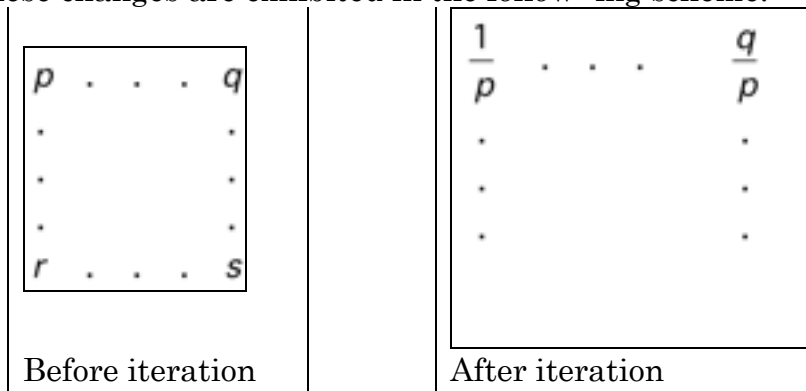
1. Pivot Entries

a) In the pivot row, the pivot element is replaced by its reciprocal; each remaining entry is the old entry divided by the pivot element.

b) In the pivot column, each old entry (except the pivot) is divided by the negative of the pivot to obtain the new entry.

2. Entries not changed in step 1.

a) Add to each entry s : the product of the old entry in the same column to the right or left of the pivot and the new entry in the same row above or below the pivot. These changes are exhibited in the following scheme:



To make sure these rules are well understood, let us carry out the operations on the condensed Tableau 5.11(b):

1. Pivot Entry. The pivot entry is 30, the pivot row is the first row and the pivot column is the first column.

a) $(p \rightarrow 1/p): 30 \rightarrow 1/30$

$(q \rightarrow q/p): 12 \rightarrow 12/30 = 2/5$

$6000 \rightarrow 6000/30 = 200$

$$\begin{aligned}
\text{b) } (r \rightarrow -r/p): \quad & 10 \rightarrow -10/30 = -1/3 \\
& 4 \rightarrow -4/30 = -2/15 \\
& -4.5 \rightarrow 4.5/30 = 3/20
\end{aligned}$$

At this point, the new condensed tableau has the form

$$\begin{array}{ccc}
1/30 & 2/3 & 200 \\
- 1/3 & \text{---} & \text{---} \\
- 2/15 & \text{---} & \text{---} \\
3/20 & \text{---} & \text{---}
\end{array}$$

2. The remaining entries: $s \rightarrow s + q(-r/p)$

s	$s + [q(-r/p)]$	
8 (row 2, column 2)	$8 + [12 (-1/3)]$	= 4
2600	$2600 + [6000 (-1/3)]$	= 600
8 (row 3, column 2)	$8 + [12 (-2/15)]$	= $32/5$
2000	$2000 + [6000 (-2/15)]$	= 1200
-4	$-4 + [12 (3/20)]$	= $-11/5$
0	$0 + [6000 (3/20)]$	= 900

Check that these computations produce Tableau 5.12(b).

Once mastered, these transformations greatly reduce the time and space necessary to use the simplex method to solve LP problems. In addition to its computational advantages, the condensed tableau also sheds light on another important topic in linear programming, *duality*. This is the subject of the next section.

E. Duality

In our continuing saga of Henry Brewster and his cheese company, I must tell you now that Henry ages all the cheeses himself from milk produced by cows on his uncle's farm. Instead of packaging his cheeses into gift assortments, Henry has the option of selling his stock to a wholesale distributor.

Suppose the distributor is willing to pay u , v , and w dollars per ounce for Cheddar, Swiss, and Brie cheeses, respectively. Then the distributor would have to offer a total of

$$m = 6000u + 2600v + 2000w$$

dollars to buy Brewster's entire stock. The distributor would like to keep the number m as small as possible.

Now Henry is not going to sell his stock if the prices offered by the distributor are too low. He quickly calculates that the distributor would pay him

$$30u + 10v + 4w$$

dollars for exactly the combination of cheeses that go into one package of the Fancy Assortment. Since Henry can receive \$4.50 for such a package if he sells it himself, he will not give up the ingredients for less than \$4.50. Thus the distributor, in his quest to minimize m , is faced with the constraint

$$30u + 10v + 4w \geq 4.5.$$

Similarly, the distributor must offer at least \$4 for the packet of ingredients that go into one box of the Deluxe Assortment; that is,

$$12u + 8v + 8w \geq 4.$$

The Distributor's Problem can then be stated as: Find nonnegative numbers u , v , and w such that

$$m = 6000u + 2600v + 2000w \text{ is minimized}$$

subject to the constraints:

$$30u + 10v + 4w \geq 4.5,$$

$$12u + 8v + 8w \geq 4.$$

Henry's problem and the distributor's problem are closely related. One is a maximization problem and the other is a minimization one. The coefficients of the function to be optimized in one problem are the constant terms in the constraints of the other problem. In fact, every coefficient appearing in Henry's problem also appears in the distributor's problem and vice-versa (see Section I.C.). These two problems are examples of what are called *dual* problems. Note that Henry's problem has two variables and three constraints, while the distributor's problem features three variables and two constraints.

We will now give the general definition of a *primal* LP problem and its *dual*.

DEFINITION Given a linear programming problem in the form: Find an $n \times 1$ vector $\mathbf{x} \geq \mathbf{0}$ such that

$$M = \mathbf{c} \cdot \mathbf{x} \text{ is maximized} \quad (\mathbf{P})$$

subject to the constraints:

$$Ax \leq b,$$

where A is an $m \times n$ matrix, b is an $m \times 1$ vector and c is a $1 \times n$ vector, the problem:

Find a $1 \times m$ vector $y \geq 0$ such that

$$m = y \bullet b \text{ is minimized} \quad (D)$$

subject to the constraints

$$yA \geq c$$

is called the *dual* of (P) and (P) is called the *primal* problem.

In expanded form, the primal problem has the familiar form: Maximize

$$M$$

subject to:

$$\begin{aligned} a_{11}x_1 \\ a_{21}x_1 \\ \dots \\ a_{m1}x_1 \end{aligned}$$

while the dual problem has the form:

$$\text{Minimize } m = b_1y_1 + b_2y_2 + \dots + b_my_m$$

subject to:

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m &\geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m &\geq c_2 \\ a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m &\geq c_n. \end{aligned}$$

The expanded form of the primal and dual problems makes clear some of the relations between them:

- a) The coefficients of the function to be maximized in the primal problem are the constants in the constraints of the dual problem;
- b) The coefficients of the function to be minimized in the dual problem are the constants in the constraints of the primal problem;
- c) A variable of the dual problem is assigned to each constraint of the primal problem;
- d) A variable of the primal problem is assigned to each constraint of the dual problem.

The dual problem is not in the standard form of a Linear Programming problem as given in Section I. It is, however, equivalent to the problem:

$$\text{Maximize } -m = -b_1y_1 + -b_2y_2 + \dots + -b_my_m$$

subject to:

$$\begin{aligned} -a_{11}y_1 + -a_{21}y_2 + \dots + -a_{m1}y_m &\leq -c_1 \\ -a_{12}y_1 + -a_{22}y_2 + \dots + -a_{m2}y_m &\leq -c_2 \end{aligned}$$

$$-a_{1n}y_1 + -a_{2n}y_2 + \dots + -a_{mn}y_m \leq -c_n.$$

A more compact form of the dual problem can be given by introducing the idea of the *transpose* of a matrix. If A is any $m \times n$ matrix, then the *transpose* of A , written A^T , is the $n \times m$ matrix whose ij th entry is the ji th entry of A ; that is, $(A^T)_{ij} = A_{ji}$.

As an example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ while the transpose

of $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is $\mathbf{b}^T = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$.

There are several important properties of the transpose which are easily proved. They are collected in the following theorem.

THEOREM 6 Let A be any $m \times n$ matrix. Then:

- a) $(-A)^T = -(A^T)$
- b) $(A^T)^T = A$
- c) $(AB)^T = B^T A^T$.
- d) If A and B are of the same size, then $A \leq B$ if and only if $A^T \leq B^T$. The proof of this theorem is left as an exercise.

Using the transpose, we may write the dual of Eq. (P) in the form: Find $\mathbf{y} \geq 0$ such that

$$M = (-\mathbf{b})^T (\mathbf{y})^T \text{ is maximized} \tag{D'}$$

subject to

$$-A^T (\mathbf{y})^T \leq (-\mathbf{c})^T.$$

This formulation is helpful in establishing the important result of the next theorem.

THEOREM 7 The dual of the dual is the primal.

Proof The primal problem is given by Eq. (P). Let the dual problem be described by Eq. (D'). Then by our first definition, the dual of Eq. (D') is:

Find a $1 \times n$ vector $\mathbf{z} \geq \mathbf{0}$ such that

$$m = \mathbf{z} (-\mathbf{c})^T \text{ is minimized}$$

subject to:

$$\mathbf{z}(-A^T) \geq (-\mathbf{b})^T.$$

This is equivalent to the problem:

Find $\mathbf{z} \geq \mathbf{0}$ such that

$$M = -m = \mathbf{Z}(\mathbf{c})^T \text{ is maximized} \tag{27}$$

subject to

$$\mathbf{z}A^T \leq (\mathbf{b})^T.$$

Now let $\mathbf{x} = (\mathbf{z})^T$ so that $(\mathbf{x})^T = ((\mathbf{z})^T)^T = \mathbf{z}$. This gives $\mathbf{z}(\mathbf{c})^T = (\mathbf{x})^T(\mathbf{c})^T = (\mathbf{c}\bullet\mathbf{x})^T = (\mathbf{c}\bullet\mathbf{x})$ since $\mathbf{c}\bullet\mathbf{x}$ is a real number. We also have $\mathbf{z}A^T = (\mathbf{x})^T(A^T) = (A\mathbf{x})^T$ so that the constraints $\mathbf{z}A^T \leq (\mathbf{b})^T$ can be written as $(A\mathbf{x})^T \leq (\mathbf{b})^T$. By Parts (b) and (d) of Theorem 6, this inequality is the same as $A\mathbf{x} \leq \mathbf{b}$. Substituting in (27), we find that the dual of the dual is: Find $\mathbf{x} \geq \mathbf{0}$ such that

$$M = \mathbf{c}\bullet\mathbf{x} \text{ is maximized}$$

subject to:

$$A\mathbf{x} \leq \mathbf{b},$$

which is the primal problem. This completes the proof.

Eventually, we wish to establish the Fundamental Duality Theorem which asserts, in part, that if the primal and dual problems both have feasible solutions, then they both have optimal solutions with $M = m$. As a step in that direction, we first prove the following Theorem.

THEOREM 8 Let \mathbf{x} be a feasible solution to the primal problem and \mathbf{y} a feasible solution to the dual. Then $\mathbf{c}\bullet\mathbf{x} \leq \mathbf{y}\bullet\mathbf{b}$.

Proof Since \mathbf{x} is feasible for the primal problem, we must have $A\mathbf{x} \leq \mathbf{b}$. We know that $\mathbf{y} \geq \mathbf{0}$, so that $\mathbf{y}(A\mathbf{x}) \leq \mathbf{y}\bullet\mathbf{b}$. On the other hand, \mathbf{y} is feasible for the dual problem so that $\mathbf{y}A \geq \mathbf{c}$, and multiplication of each side by the nonnegative vector \mathbf{x} gives $(\mathbf{y}A)\mathbf{x} \geq \mathbf{c}\bullet\mathbf{x}$. We combine the inequalities with the associative rule for matrix multiplication to obtain:

$$\mathbf{c}\bullet\mathbf{x} \leq (\mathbf{y}A)\mathbf{x} = \mathbf{y}(A\mathbf{x}) \leq \mathbf{y}\bullet\mathbf{b} \text{ which is the desired result.}$$

Corollary 1 If the primal and dual problems both have feasible solutions, then both have optimal feasible solutions.

Proof Let \mathbf{y} be any feasible solution to the dual problem. The primal problem fails to have an optimal solution exactly when the function $M = \mathbf{c}\bullet\mathbf{x}$ is unbounded. But for any feasible solution \mathbf{x} of the primal problem, Theorem 8 asserts that M is no larger than the fixed number $\mathbf{y}\bullet\mathbf{b}$. Thus M is bounded and the primal problem must have an optimal, feasible solution. The proof that the dual problem also has an optimal, feasible solution is similar.

Corollary 2 Suppose \mathbf{x} is a feasible solution to the primal problem and \mathbf{y} is a feasible solution of the dual problem. If $\mathbf{c}\bullet\mathbf{x} = \mathbf{y}\bullet\mathbf{b}$, then \mathbf{x} and \mathbf{y} are optimal solutions of the primal and dual problems, respectively.

Proof Suppose, to the contrary, that \mathbf{x}^* is a feasible solution to the primal problem with $\mathbf{c}\bullet\mathbf{x}^* > \mathbf{c}\bullet\mathbf{x}$. Since $\mathbf{c}\bullet\mathbf{x} = \mathbf{y}\bullet\mathbf{b}$, we have $\mathbf{c}\bullet\mathbf{x}^* > \mathbf{y}\bullet\mathbf{b}$, contradicting Theorem 8.

Suppose, then, that we have feasible solutions to the primal and dual problems. According to Theorem 8 and its corollaries, both problems have optimal, feasible solutions. Suppose M^* is the largest feasible value for M in the primal problem and m^* is the smallest feasible value for m in the dual problem. From Theorem 8, we have $M^* \leq m^*$. In fact, it turns out that $M^* = m^*$. This statement is part of the Fundamental Duality Theorem. We will show how the simplex method can be used to prove this statement.

We may begin by examining the condensed initial tableaux for the primal and dual problems. These will have the form of Tableaux 5.13 and 5.14.

Tableau 5.13 Initial condensed tableau for primal

x_1	x_2	...	x_n	
a_{11}	a_{12}	...	a_{1n}	b_1
a_{21}	a_{22}	...	a_{2n}	b_2
...				
a_{m1}	a_{m2}		a_{mn}	b_m
...				
$-c_1$	$-c_2$...	$-c_n$	0

Tableau 5.14 Initial condensed tableau for dual

y_1	y_2	...	y_m	
$-a_{11}$	$-a_{21}$...	$-a_{m1}$	$-c_1$
$-a_{12}$	$-a_{22}$...	$-a_{m2}$	$-c_2$
...				
$-a_{1n}$	$-a_{2n}$		$-a_{mn}$	$-c_n$
...				
b_1	b_2	...	b_m	0

The simplex method proceeds by selecting a sequence of pivot elements and using them to perform Gauss-Jordan eliminations. We wish to show that if corresponding pivot entries are chosen in the tableaux for the primal and dual problems at every stage, then the pattern of relations among the coefficients evidenced in Tableaux 5.13 and 5.14 is retained after the Gauss-

Jordan process has been completed. To see why this is true, suppose at some stage, the primal and dual tableaux look like Tableaux 5.15 and 5.16.

Tableau 5.15 Primal problem
before iteration

.
.
...	$[p]$...	q	...
.
...	r	...	s	...
.
.
...	y	...	y'	...
				M

Tableau 5.16 Dual problem
before iteration

.
.
...	$[-p]$...	$-r$...
.
...	$-q$...	$-s$...
.
.
...	x	...	x'	...
				$+m$

We now perform the basic iteration step of the simplex method using p for the Pivot element in the primal problem and $-p$ in the dual problem. Using the rules of Section III.D, we obtain the post-iteration condensed tableaux: Tableaux 5.17 and 5.18.

Tableau 5.17 Primal problem
after iteration

.
.
...	$1/p$...	q/p	...
.
...	$-r/p$...	$s-(qr/p)$...
.
.
...	$-y/p$...	$y'(qy/p)$...
				$M-(xy/p)$

Tableau 5.18 Dual problem
after iteration

.
.
...	$-1/p$...	$-r/p$...
.
...	$-q/p$...	$-s+(rq/p)$...
.
.
...	x/p	...	$x+(-xr/p)$...
				$+m+(xy/p)$

A comparison of the tableaux shows that the duality pattern which existed before iteration continues to hold after iteration. Note also that the sum of the numbers in the lower right-hand corner of the primal and dual tableaux has been retained; it is equal to $M + m$ both before and after the iteration. In particular, if both M and m start at 0, so that $M = -m$, this inequality will be maintained at each and every step.

What happens when we reach the optimal, feasible solution? In the primal tableau, the entries in the last row will be nonnegative. The entries in the last column will also be nonnegative because they represent values of the basic variables. But the entries in the last column ($x/p, \dots, x'-(xr/p), \dots$) of the primal tableau are exactly the entries in the last row of the dual tableau. (See Tableaux 5.17 and 5.18.) Since all these entries are nonnegative, we have reached an optimal solution of the dual problem also. The optimal values of

both problems are given by the numbers in the lower right-hand corner of the tableaux. But we have just shown that their absolute values are equal! Thus, the maximum value of M in the primal problem. is exactly the same as the minimum value of m in the dual problem. This is the result we promised.

We actually can deliver more than we promised. The solution of the dual problem would appear in the last column of the final tableau for the dual problem. But the entries in this column ($-y/p,$.., $y' -(qy/p),$..) also appear in the last row of the final tableau for the primal problem. (See Tableaux 5.17 and 5.18.) Thus, the solution of the dual problem can be read off from the solution tableau of the primal problem. By duality, we can turn things around; that is, the solution of the primal problem can be read off the final tableau for the dual. In summary, the solution to both the primal and dual problems is contained in the final tableau of either problem.

Let us illustrate some of these results with the Fromage Cheese Company example. The variables x and y represented the number of packages of the two assortments to be prepared. The variables u , v , and w were slack variables in the original primal problem corresponding to excess amounts of Cheddar, Swiss, and Brie cheeses. In the dual problem, they represented the prices to be offered by the distributor for an ounce of each of these cheeses. The final tableau for the primal problem is shown in Tableau 5.6.

Tableau 5.6

x	y	u	v	w	M		
1	0	0	1/6	-1/6	0	100	x
0	1	0	-1/12	5/24	0	200	y
0	0	1	-4	5/2	0	600	u
0	0	0	5/12	1/12	1	1250	M

We read off the solution of the distributor's problem from the bottom row:

$$u = 0, v = 5/12, w = 1/12, M = 1250.$$

The distributor's best offer to Brewster is that he will buy all the cheese at \$0 per ounce of Cheddar, \$5/12 per ounce of Swiss, and \$1/12 per ounce of Brie. This will cost the distributor \$1250, exactly the same amount that Henry would earn if he sold all the cheese in gift packages. The role of x and y in the distributor's problem is to measure slack, the amount that the distributor's offer exceeds the constraints, $30u + 10v + 4w \geq 4.5$ and $12u + 8v$

$+ 8w \geq 4$. Since $x = y = 0$ in the solution of the dual problem, there is no slack. The distributor's offer exactly meets the constraints.

We may summarize what we have proved in this section by stating the central theorem about duality.

THEOREM 9 (FUNDAMENTAL THEOREM OF DUALITY) Let Eq. (P) be a primal linear programming problem and Eq. (D) its dual.

- a) If both problems have feasible solutions, then both have optimal, feasible solutions and the maximum value of M is the same as the minimum value of m .
- b) If one of the problems has feasible solutions, but the objective function (M or m) is unbounded, then the other problem has no feasible solutions.
- c) If one of the problems has a feasible solution, but the other problem does not, then the problem with feasible solutions has no optimal feasible solution.
- d) It is possible that neither problem has a feasible solution.

Proof a) We have already indicated how the simplex method may be used to prove this result;

b) The proof is by contradiction and use of Corollary 1 of Theorem 8;

c) The proof is left as an exercise for the reader;

d) Consider the primal problem of maximizing $x_1 + x_2$ subject to the constraints $x_1 - x_2 \leq -1$ and $-x_1 + x_2 \leq 0$. It is easy to show that the feasibility set of this problem and of its dual are both empty.

There are many other interesting aspects of duality theory. You may find extended discussions of these in any of the more advanced texts on linear programming listed in the References at the end of this chapter. I do want to indicate here, however, several applications of duality.

In the first place, the natural formulation of a particular question may be a linear programming problem in dual form; for example, the problem of the distributor buying cheese from Henry Brewster. To solve this problem directly, using the simplex method, requires the addition of both slack variables and artificial variables before an initial basic feasible solution can be found. If we switch to the primal problem, less work is required as a basic feasible solution is produced as soon as slack variables are introduced.

As another example, suppose that an optimal feasible solution of a primal LP problem has been found. It is then discovered that an additional constraint should have been used but was omitted from the original formulation of the problem. When this constraint is added, the feasibility set may be significantly affected. The optimal feasible solution of the original problem may not even be feasible for the new problem. If we restrict ourselves to using only primal problems, then we would have to start all over again. The addition of a constraint in the primal problem, however, corresponds to the addition of a new variable in the dual problem. The basic feasible optimal solution of the original problem gives a basic feasible solution to the dual problem even with the new variable added (simply let the new variable be nonbasic) and the simplex method can continue from this step; instead of going back to $M = 0$, we are beginning from a relatively large value of M .

As a final example, duality theory has a very close connection to the theory of two-person games. The Fundamental Duality Theorem and the simplex method of linear programming provide a constructive proof of the best known result in two-person zero-sum game theory, the Von Neumann Minimax Theorem.

F. Some Wrinkles in the Simplex Method

In the basic iteration step of the simplex method of solving linear programming problems, the first task is to select the pivot element. The pivot column is determined by choosing a column whose entry in the bottom row of the tableau is negative. Any such column will do, although in practice we usually select the column by finding the negative entry in the bottom row with greatest absolute value.

Once the pivot column is chosen, the pivot row is selected by examining the ratios of the entries in the last column of the tableau to the corresponding positive entries in the pivot column. The pivot row is the row for which these ratios is smallest.

We have already discussed what happens if all the entries in the pivot column are nonpositive (Section III.B). In such a case, the function M is unbounded and the LP problem has no optimal, feasible solution.

Suppose that some of the entries in the pivot column are positive, but that there is a "tie" between two or more of the ratios for the minimum value. In this case, we can choose any of the associated rows as the pivot row. After applying the Gauss-Jordan procedure, two (or more) of the variables that

were in the basis will have become zero. However, only one of them has left the basis (the one determined by the pivot row); the others remain in the basis. In such a case, it is possible that the iteration does not increase the value of M ; it may remain unchanged.

There is another situation in which the iteration does not improve the value of M . Suppose that a particular entry in the pivot column is positive, but the entry in the last column of the tableau in the associated row is 0. This will happen only if one of the basic variables has value 0 at this stage. Then the minimum ratio will be 0 so that this variable will be picked to leave the basis. The variable that enters the basis will remain equal to 0 and the value of M will not change when the Gauss-Jordan procedure is implemented. Geometrically, we will not have moved to a new vertex by this iteration, but have only changed our minds about which variables to call "basic."

The term *degeneracy* is used to indicate a situation when we have a basic feasible solution in which at least one of the basic variables has value 0. Degeneracy occurs fairly frequently in the solution of linear programming problems, but it commonly causes no difficulty. The simplex algorithm may be continued and after several more steps, the value of M will begin to increase again.

It is theoretically possible, however, that the value of M will never increase, but that we will cycle repeatedly through a set of nonoptimal basic feasible solutions. The simplex method, as we have described it, cannot prevent this from happening. Imagine for a moment a feasibility set in which vertices $V3$, $V7$, and $V2$ are all adjacent to each other, but the optimal value of M occurs at vertex $V1$. It is possible that the simplex algorithm will take us from $V3$ to $V7$ to $V2$ and then back to $V3$ again. If this occurs, we will repeat these steps over and over and never reach the optimal vertex $V1$.

Artificially constructed examples of cycling under the simplex method have been discovered, so the theoretical possibility is really an actual one. Oddly enough, cycling has never occurred in any of the thousands of linear programming problems arising from real-world situations that have been solved by the simplex method. It is comforting to know, however, that the simplex method can be modified to avoid the possibility of cycling.