II CONVEX SETS

A. DEFINITION, EXAMPLES, BASIC PROPERTIES

B. POLYGONAL CONVEX SETS IN THE PLANE

Linear Programming Part II

Industrial production, the flow of resources in the economy, the exertion of military effort in a war theater-all are complexes of numerous interrelated activities. Differences may exist in the goals to be achieved, the particular processes involved, and the magnitude of effort. Nevertheless, it is possible to abstract the underlying essential similarities in the management of these seemingly disparate systems.

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II. CONVEX SETS

A. Definitions, Examples, Basic Properties

In Part II, we will discuss in more detail the nature of the feasibility set of a linear programming problem. We will concentrate on feasibility sets arising from problems involving two variables. In such cases, the feasibility set lies in the plane and we can make use of geometric intuition. Our aim is to outline a proof that optimal feasible solutions of LP problems can be found at the vertices of the feasibility sets.

DEFINITION If x = $(x_1 x_2,\ldots,x_n)$ **and y =** $(y_1 y_2,\ldots,y_n)$ **are points in Euclidean n-dimensional space, then the *line segment* between x and y is the set of vectors of the form**

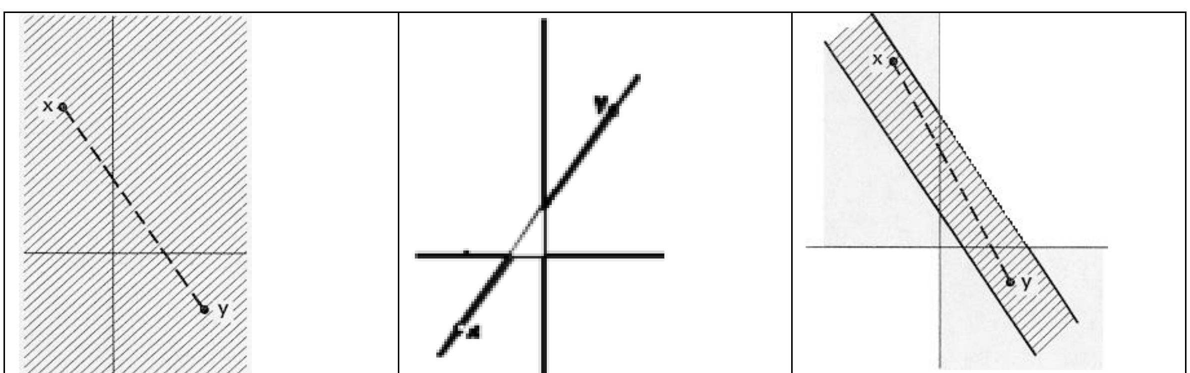
$$\boldsymbol{w} = t\mathbf{x} + (1- t)\,\mathbf{y} \ \ \textbf{where } 0 \le t \le 1 \qquad (10)$$

You may think of the parameter t in Eq. (10) as a time variable and consider that the line segment is traced out by moving from **y** at $t = 0$ to **x** at $t = 1$. The set of all vectors satisfying Eq. (10) where t can be any real number is the set of points of the entire line through **x** and **y**.

DEFINITION A subset *K* of Euclidean *n*-dimensional space is said to be *convex* if, whenever x and *y* belong to *K,* then so does every point of the line segment between x and *y.*

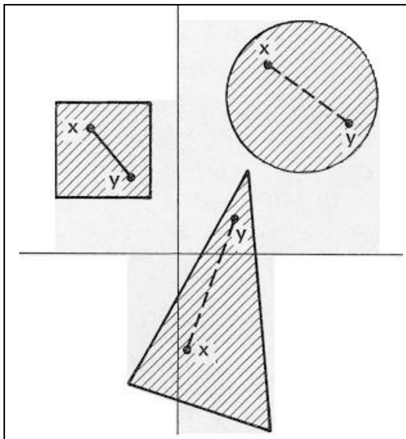
The following are all examples of convex figures in the plane (see Fig. 5.5)

a) The entire plane,
b) A straight line in the plane,
c) The region between two parallel lines,
d) The interior of a square, triangle, or circle,
e) The first quadrant of the plane.

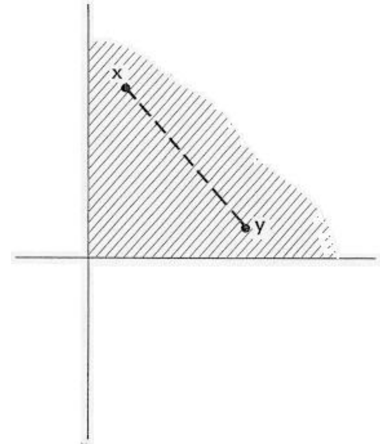


(a) The plane | (b) A line | (c) Region between

		parallel lines



(d) Circles, squares, or triangles



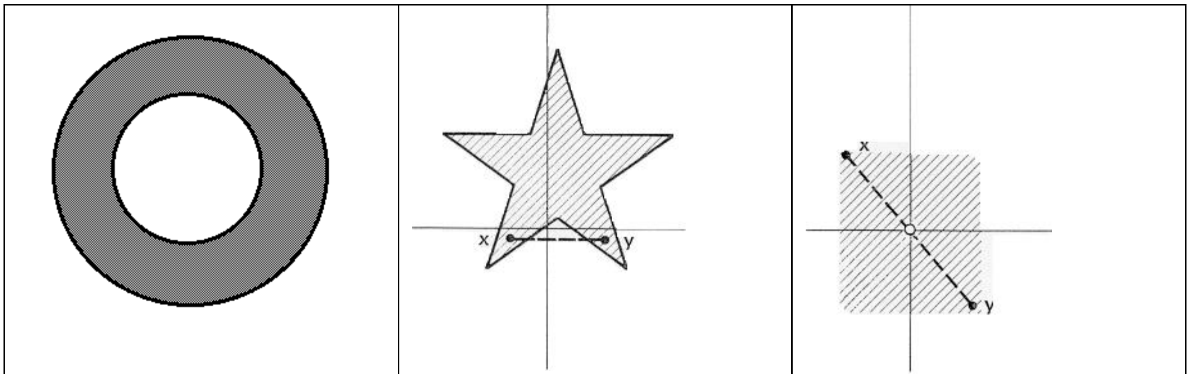
(e) The first quadrant

Fig. 5.5 Convex sets.

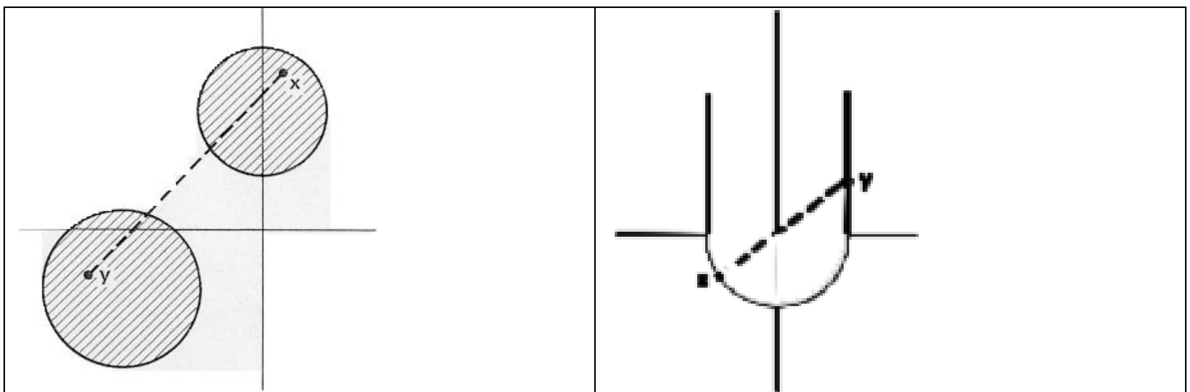
The idea of a convex set can be further clarified by examining some sets in the plane which are not convex (see Fig. 5.6):

a) The region between two concentric circles,
b) The interior of a star,
c) The plane with the origin deleted,
d) Two disjoint disks,
e) The letter U.

The pictures of Fig. 5.6 indicate that it is often easy to prove that a particular set is not convex. We just have to locate two points in the figure so that the straight line segment between them does not lie entirely in the figure.



(a) Region between concentric circles | (b) Interior of a star | (c) Plane with origin (0,0) deleted



(d) Two disjoint disks | (e) The letter *U*

Fig. 5.6 Nonconvex sets. In each case, the segment between x and y does not lie entirely inside the set.

To establish the convexity of a figure usually requires more work. We shall see how this is done for a particular set in Theorem 2. First, we give an important definition.

DEFINITION If a is a given $1 \times n$ vector and b a given constant, then the set of all vectors x in Euclidean *n*-dimensional space satisfying a $\bullet x \le b$ is called a *closed half-space.* The set of vectors for which a $\bullet$ x = b is called the *boundary* of the closed half-space.

In the special case $n = 2$, the inequality $a_1x_1 + a_2x_2 \le b$ is satisfied by exactly those points which lie on one side of the boundary line $a_1x_1 + a_2x_2 = b$.

THEOREM 2 Every closed half-space is a convex set.

Proof Suppose **y** and **z** lie in the closed half-space consisting of vectors which satisfy a $\bullet\mathbf{x} \leq b$. Let w be any point on the line segment between **y** and **z**. Then we have
a $\bullet$y $\leq b$, a $\bullet$ z $\leq b$, and $\boldsymbol{w} = t\mathbf{y} + (1 - t)\mathbf{z}$ for some t in [0, 1].

We must show **that a** $\bullet\boldsymbol{w} \leq b$.

Now

$\mathbf{a} \bullet \boldsymbol{w} = \mathrm{a} \bullet (t\mathbf{y} + (1 - t)\mathbf{z})$
$= t(\mathbf{a} \bullet \mathbf{y}) + (1 - t)(\mathbf{a} \bullet \mathbf{z})$
$\leq t\mathbf{b} + (1 - t)\mathbf{b}$ since t and $1 - t$ are nonnegative
$= \mathbf{b}$

The next theorem gives a very general and very important property of convex sets.

THEOREM 3 The intersection of any collection of convex sets is convex

Proof: Let *P* and *Q* be any two points in the intersection. Then *P* and *Q* belong to each convex set of the collection. But each convex set contains the segment between *P* and *Q*. Thus the segment belongs to every set in the collection so that it belongs to the intersection. 0

Theorems 2 and 3 provide the first essential fact about feasibility sets. Each feasibility set of a linear programming problem consists of all vectors that simultaneously satisfy a finite number of linear constraints. Each constraint defines a closed half-space. Thus the feasibility set is the intersection of a finite number of closed half-spaces, each of which is convex by Theorem 2. Theorem 3 then gives us

THEOREM **4** The feasibility set of a linear programming problem is a convex set.

DEFINITION A *polygonal convex set* is the intersection of a finite number of closed half-spaces.

B. Polygonal Convex Sets in the Plane

Now let us focus attention on polygonal convex sets in the plane. An *edge* of a polygonal convex set *K* is defined to be the intersection of *K* with the boundary line of a closed half-plane determining *K*. Since an edge is the intersection of two convex sets, it must also be convex. In fact, it is a convex subset of a line. There are only a few possibilities for the geometric character of an edge.

***THEOREM* 5** A subset K of a line is convex if and only if K is one of the following:

a) K is the entire line;
b) K is the empty set;
c) K is an open or closed ray;
d) K is a segment of the line, with or without either endpoint; or
e) K is a single point.

Proof: It is easy to verify that each of these sets of type (a)-(e) is convex. We shall show how to prove the converse. Suppose K is a convex subset of the line. If K is nonempty, then there is at least one point p of the line that belongs to K. If K is not the entire line, then there is at least one point q of the line which does not belong to K.

Now we can parameterize the line so that it consists of all points of the form

$w_t = tp + (1 - t)q$ where t can be any real number.

The *positive side* of the line consisting of all points w_t for which t is positive and the *negative side* is similarly defined. We are given that p is on the positive side of q. Since K is convex, and q does not belong to K, there can be no points of K on the negative side of q. All points of K lie on the positive side of q.

Thus the set of parameter values t corresponding to points of K is bounded below by O. Since this is a set of real numbers, it has a greatest lower bound t_r with $t_r \geq 0$. Let r be the corresponding point of the line; that is $r = t_r$ (It is possible that $r = q$.) By the definition of greatest lower bound and the fact that K is convex, we have that no point to the left of r belongs to K and all points between, r and p belong to K.

Now if all the points to the right of r (that is, all points w_t with $t > t_r$) belong to K, then K is either a closed or open ray, depending on whether or not r belongs to K. Suppose then that some point s to the right of r does not belong to K. Let the corresponding parameter value be t_s.

Consider again the set of parameter values t corresponding to points of K. This set is bounded above by t_s and so it must have a least upper bound t_u. Let u be the corresponding point of the line, u . Then K consists of all points between r and u, including or excluding the points r and u. Thus if K is a convex subset of a line and is not of type (a), (b), (c), or (e), then it must be of type (d), See Fig. 5.7. 0

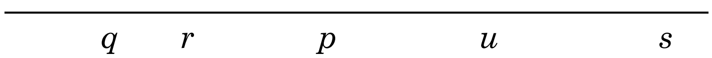


Fig. 5.7 Location of points along K as given in proof of Theorem 5.

Not all of the cases mentioned above can occur for the feasibility set of an **LP** problem. Recall that an edge of a polygonal convex set in the plane is the intersection of a boundary line of a closed half-space with the set. Because of this, it can be shown, by reasoning similar to that in the proof of Theorem 5, that an edge containing more than one point must be either a closed segment or a closed ray; that is, an edge always contains its endpoints.

By a *vertex* of a polygonal convex set K in the plane we will mean a point of K which is contained in at least two distinct boundary lines. A vertex will be an endpoint of the edge of the polygonal convex set. We want to prove the fundamental theorem that a linear function defined on K assumes its largest and smallest values at vertex points.

First, consider a linear function $f(x, y) = \alpha x + \beta y$ defined along some line L with equation $y = mx + b$. Then we have

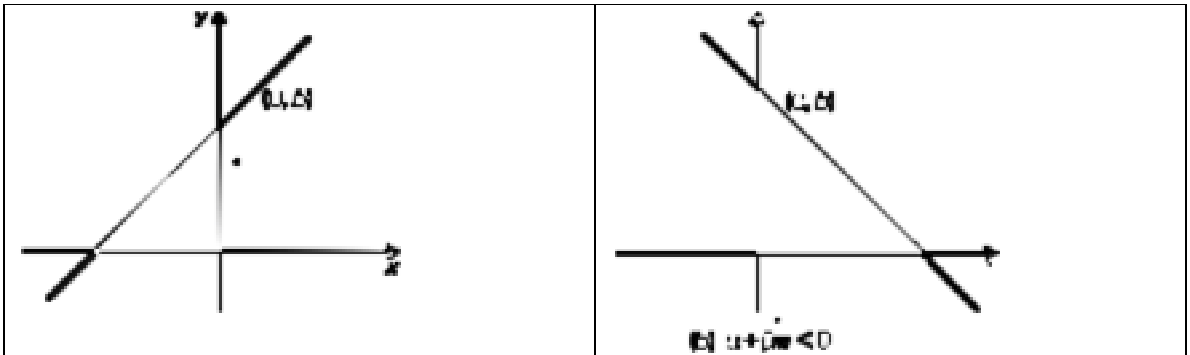
$$f(x, y) = \alpha x + \beta\,(mx + b) = (\alpha + \beta M)x + \beta b.$$

If the quantity $\alpha + \beta M$ is zero, then the function is constant along L. If $\alpha + \beta M$ is positive, then f is a strictly increasing function of x, while if $\alpha + \beta M$ is negative, f is a strictly decreasing function of x (see Fig. 5.8).

In any case, if we examine the function along some closed segment of the line L, then the minimum value of f will occur at one endpoint and the maximum values at the other. If we examine f along a closed ray, then there are two possibilities:

1. The minimum value occurs at the endpoint of the ray and there is no maximum value for f, or

2. The maximum value of f occurs at the endpoint and there is no minimum



$\alpha + \beta M > 0$	$\alpha + \beta M < 0$

Fig. 5.8 The graph of $y = (\alpha + \beta M)x + \beta b$.

We are now ready to outline the proof of the basic theorem on the location of extreme values of a linear function at the vertices of a polygonal convex set.

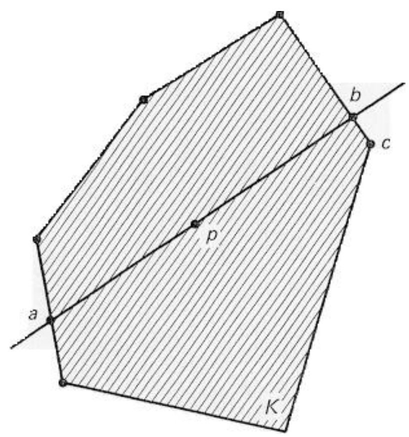


Fig. 5.9 Intersection of a line and a bounded polygonal convex set in the plane.

Suppose K is a polygonal convex set in the plane and that p is an interior point of K; that is, p is not on any edge. See Fig. 5.9. Let L be any line containing p and suppose L intersects the boundary of K in two points. Then the value of f at p must lie between the values of f at these two points. We may label the two points a and b in such a way that

$$f(a) \leq f(p) \leq f(b). \quad (11)$$

Now the point b lies on an edge of K with vertices c and d. On this edge, the function f takes on its extreme values at the vertices. Label the vertices so that

$$f(c) \leq f(b) \leq f(d). \quad (12)$$

Combining inequalities (11) and (12) gives us $f(p) \leq f(d)$. In other words, associated with each interior point of the convex set K, there is a vertex at which the value of f is at least as large.

Similarly, we can find a vertex e of the edge containing a with $f(e) \leq f(a) \leq f(p)$, so that given an interior point p of K, there is a vertex at which f is at least as small.

Since a polygonal convex set has a finite number of vertices, there are vertices at which the function f assumes its greatest and smallest values. This completes an outline of the proof of the desired result. This is only an outline, because we have not considered all possibilities. We assumed, for example, that the line L through p intersected the boundary of the convex set in exactly 2 points. It may happen that the line does not intersect the boundary at all or that it intersects the boundary in only one point. Figure 5.10 illustrates these possibilities.

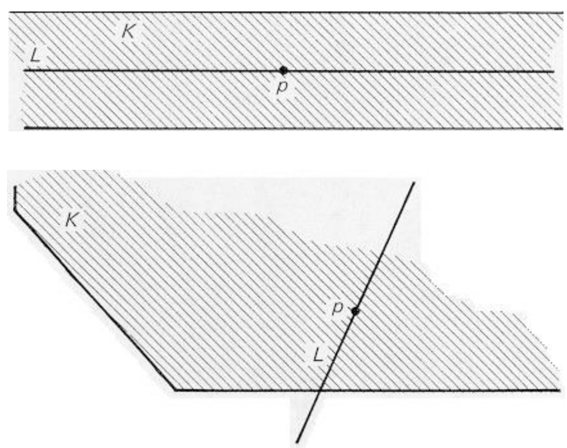


Fig. 5.10 Two possible ways a line might intersect an unbounded polygonal convex set in the plane.

In the case where the line and the boundary do not intersect, there is no problem. Either the function f is constant on the line or it takes on all real values. If the latter occurs, there is no optimal solution to the LP problem.

Suppose the line intersects the boundary at one point. If f has no maximum on the line, the LP problem again has no optimal solution. On the other hand, if f has a maximum, it must occur at the point on the boundary. That boundary point lies on an edge of K. If the edge is a closed segment, then f assumes a value at one of the two vertices of that edge which is greater than the value $f(p)$. If the edge is a closed ray, then again there is either no maximum for f or the value of f at the endpoint of the ray exceeds the value $f(p)$.

In every case, then, if f takes on a maximum value on the polygonal set K, it takes that value at one of the vertices.

To make use of simple geometric figures in the plane, we have restricted our considerations to LP problems involving only two variables.

Even in this case, we have presented only an outline of the proof that the optimal feasible solution occurs at a vertex point. Furthermore, our proof involved checking a fair number of special cases.

If the reasoning discussed here is generalized to LP problems involving more than two variables, it is reasonable to suspect that the number of special cases that can arise will be outrageously large. Surely, I hope you are thinking, there is a way to generalize the ideas of this proof which handles all the cases at the same time. There is indeed such a proof for the general LP problem with n variables, but it requires some mathematical tools we do not have space to develop in this book. The reader with a stronger background in linear algebra
or functions of several variables may wish to consult the proofs in the following References: George B. Dantzig, *Linear Programming and Extensions* and George Hadley, *Linear Programming*.