Linear Programming Part I

Industrial production, the flow of resources in the economy, the exertion of military effort in a war theater-all are complexes of numerous interrelated activities. Differences may exist in the goals to be achieved, the particular processes involved, and the magnitude of effort. Nevertheless, it is possible to abstract the underlying essential similarities in the management of these seemingly disparate systems.

George B. Dantzig

WHAT IS LINEAR PROGRAMMING?

A. Introduction

Many problems that arise in the real world have to do with finding the optimal values of some variables. A businessperson is usually concerned with maximizing profits, although in some situations he or she may want to minimize costs. The dietician for a large high school has the responsibility for providing a nutritionally adequate hot-lunch program that puts the smallest burden on the school budget. Transportation engineers may wish to design a mass transit system that will carry the largest number of workers from their jobs to their homes during the rush hours. Forest rangers are interested in picking locations to station fire-fighting equipment to minimize the time needed to reach any blaze that may erupt in the forest. Countless other examples of this nature may be given.

A large class of these problems can be successfully modeled as problems of optimizing a linear function of several variables subject to a finite number of linear constraints. The subject of *linear programming* (often abbreviated as LP) deals with the formulation and solution of such problems.

The formal theory of linear programming was not developed until shortly after World War **II,** although various special linear programming problems had been solved earlier. At first glance, the standard methods of calculus to locate maxima and minima of functions might seem to be all that is needed to solve such problems. Unfortunately, calculus is rarely of much help as the extreme values of a linear function almost always occur on the boundary of its domain. New techniques were required. Fortunately, in 1947 an American mathematician, George B. Dantzig, discovered an efficient algorithm for solving linear programming problems. Dantzig's algorithm and the availability of high-speed computers have made possible the application of linear programming models to many important real-world decision-making situations. In the next section, we will present a fairly simple **LP** problem and a detailed discussion of its solution. Although the example is not a very sophisticated one, it does evidence many of the important concepts that arise in linear programming.

B. A Detailed Example: Maximizing Revenue with Limited Resources

Each year the Fromage Cheese Company has a sale to celebrate the anniversary of the opening of its first store. This year, company president Henry Brewster decided to offer two gift packages of cheese at a special price.

The *Fancy Assortment* will contain 30 ounces of Cheddar cheese, 10 ounces of Swiss cheese, and 4 ounces of Brie. The *Deluxe Assortment* will be a package with 12 ounces of Cheddar, 8 ounces of Swiss, and 8 ounces of Brie. In the past, these two assortments have been very popular and Brewster is

certain that he can sellout his entire stock if he prices the Fancy Assortment at $4.50 a box and charges $4 for the Deluxe combination.

Brewster has in his storage rooms 6000 ounces of Cheddar, 2600 ounces of Swiss, and 2000 ounces of Brie. He must decide how many packages of each assortment to prepare. Being a prudent businessman, he would like to find the numbers that will maximize his revenue.

We may begin to develop a mathematical formulation of Brewster's problem by letting x denote the number of packages of the Fancy Assortment and y the number of packages of the Deluxe Assortment that will be prepared. Brewster's job is to determine the values of x and y for which the quantity $M = 4.5x + 4y$ is maximized.

The total amount of Cheddar cheese that would be used in x packages of the Fancy Assortment and y packages of the Deluxe would be $30x + 12y$. Of course, he cannot sell more Cheddar cheese than he has in stock. This implies that there is a constraint on the number of packages he may prepare; namely, x and y must be restricted so the inequality

$$30x + 12y \leq 6000 \hspace{4cm} (1)$$

holds.

There are similar constraints on the Swiss and Brie cheeses. These are, respectively,

$$10x + 8y \leq 2600 \hspace{4cm} (2)$$

and

$$4x + 8y \leq 2000 \hspace{4cm} (3)$$

Brewster's choice for the number of packages of the two assortments he can sell is governed by these three inequalities together with the trivial observation that x and y must be nonnegative numbers. Mathematically, he is restricted to the choice of a point in the *constraint* (or *feasibility) set*

C = { (x,y): $x \geq 0$, $y \geq 0$, x and y satisfy (1), (2) and (3) }.

The constraint set C can be described nicely in a geometric way. Consider the inequality (1). The points in the plane whose coordinates satisfy this inequality are the points which lie on or below the line L_1 whose equation is

$$30x + 12y = 6000. \hspace{4cm} (4)$$

Since x and y cannot be negative, Brewster is only concerned with points in the first quadrant of the (x,y)-plane which lie on or below L_1 This set is the shaded triangle shown in Fig. 5.1.

Now we may consider the effect of the second inequality. This forces Brewster to choose only those points which lie on or below the line L_2 whose equation is

$$10x + 8y = 2600. \tag{5}$$

The combined effect of inequalities (1) and (2) is to restrict Brewster's choice to the region of the first quadrant consisting of those points which lie on or below both the lines L_1 and L_2. This region is shown in Fig. 5.2.
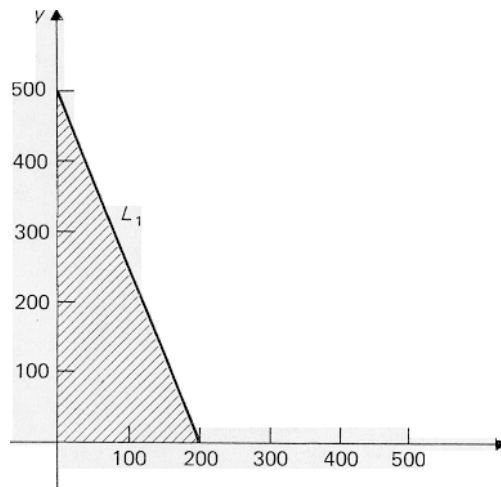

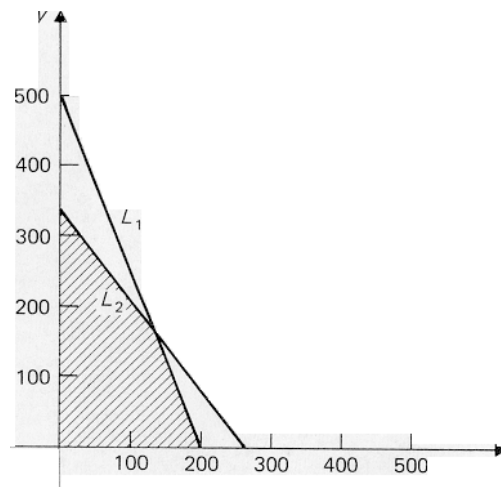**Fig. 5.1** The feasibility set determined by inequality (1).

Fig. 5.2 The feasibility set determined by inequalities (1) and (2).

The third inequality (3) imposes one final restriction. The points not only must lie in the first quadrant below the lines L_1 and L_2, they must also lie on or below the line L_3 with equation

$$4x + 8y = 1000. \tag{6}$$
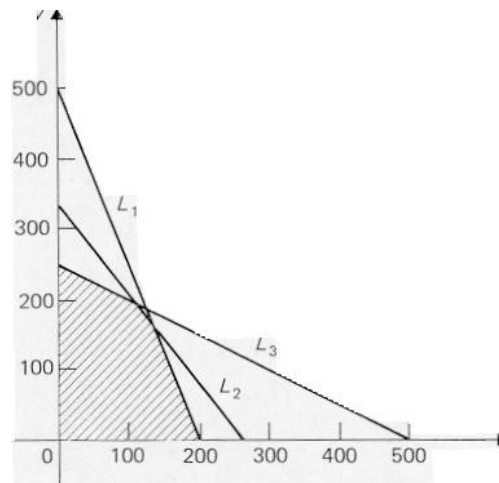
This set of points is shown in Fig. 5.3.

Fig. 5.3 The feasibility set determined by inequalities (1), (2), and (3).

We have a geometric representation of the constraint set of Brewster's problem. It is a polygonal region in the plane with five vertices: (0,250), (0,0), (200,0), (140,150), (100,200). The coordinates of the last two vertices listed here are found by determining the points of intersection of L_1 with L_2, and L_2 with L_3. This involves solving pairs of linear equations, a straightforward algebraic procedure.

The coordinates of any point in this polygonal region give a feasible mixture of the two assortments. There are, of course, still an infinite number of points in the constraint set so there are infinitely many different combinations of Deluxe and Fancy Assortments that can be packaged with the cheese that is available.

Let's see how we can narrow Brewster's choices even more. Suppose he is considering for the moment a mixture $Po = (xo, yo)$ which corresponds to a point in the interior of the constraint set. (See Fig. 5.4.)

If *Po* lies in the interior of C, then we can find a point, like $P_0 = (x_1, y_1)$, which also lies in C but both of whose coordinates are greater than the corresponding coordinates of *Po*. In other words, it is possible to squeeze out a few more packages of each cheese assortment with the available inventories. But the more packages Brewster can prepare, the more revenue he takes in. Thus, no point in the interior of the constraint set is an optimal choice for Brewster. The optimal choice will be one of the points on the boundary of C. We have helped Brewster narrow his choices to a smaller set, but there are still infinitely many different combinations from which to pick. We need some further restrictions.
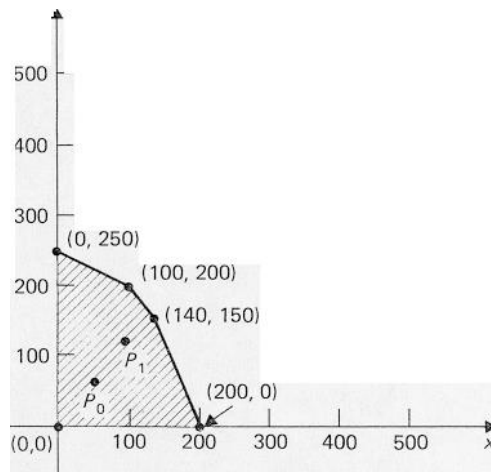

Fig. 5.4 The constraint set for the Fromage Cheese Company example.

The number Brewster is trying to maximize is his total revenue, $M = 4.50x + 4y$. Let's examine how this quantity behaves along one of the edges of the boundary of the constraint set, say the straight line segment from (0,250) to (100,200). This is a piece of the line L_3, so the coordinates of any point on this edge must satisfy equation (6). The edge can be described analytically as

$$\{ (x,y): 4x + 8y = 2000, 0 \le x \le 100\}$$

For points on this edge, we have $8y = 2000 - 4x$ so that $4y = 1000 - 2x$. Thus the revenue obtained from a point (x, y) on this edge can be written as

$$M = 4.50x + 4y = 4.50x + 1000 - 2x = 2.50x + 1000.$$

Clearly the larger we can make x, the larger we will make the revenue. But we have seen that x can be no larger than 100 if we are to remain on this edge. If Brewster is going to choose a point on this edge, then he should choose the vertex (100,200).

We can argue similarly for the other edges of the boundary. For any edge, the revenue is maximized at one of the endpoints of the edge; that is, at one of the vertices of the constraint set.

Hence, Brewster needs only to consider the five vertices of C to find his optimal mixture. His problem is reduced to deciding among a **finite** set of alternatives. One way he can do this is to list all the vertices and compute the revenue associated with each one. This is done in Table 5.1.

The table shows that the optimal choice for Brewster is to prepare 100 packages of the Fancy Assortment and 200 packages of the Deluxe Assortment. This will produce revenue of $1250. This particular mixture uses up all of the Swiss and Brie cheese he has, but only 5400 of the 6000 available ounces of Cheddar.

Table 5.1

Vertex (x,y)	Revenue = $4.5x + 4y$
(0,0)	$0
(0,250)	$1000
(100,200)	$1250
(140,150)	$1230
(200,0)	$900

The constraint set, and hence its vertices, are determined without consideration of the particular prices that will be charged for the two cheese assortments. If Brewster decides to charge amounts other than $4.50 and $4 for the two assortments, his optimal mixture will still be given by one of the five vertices we have found. To find the best mixture, he need only retest the vertices with the new revenue function. If his stock should change, or if he should decide to alter the relative proportions of the three cheeses in the assortments, then the constraint set would change. In such a case, he would have to find the vertices of the new constraint set and test the revenue function at each of these points.

What can we learn from this example that will be useful for the general linear programming problem? In the next section, we will formulate the general problem and see that its solution is qualitatively much like that of the Fromage Cheese Company example.

C. The Linear Programming Problem

The mathematical problem of the previous section – stripped of its cheesy crust – is simply this:

Maximize the quantity $M = 4.5x + 4y$

subject to the constraints:
$$30x + 12y \leq 6000,$$
$$10x + 8 \leq 2600,$$
$$4x + 8y \leq 2000,$$
$$x \geq 0, y \geq 0$$

The important feature of the expressions occurring in this problem is that all of the variable terms are of degree one; that is, x and y occur alone and only to the first power. There are no terms of type $xy,\ x^2,\ xy^3,\ x^{1/2}$, Expressions of the form $ax + by$ (a, b constants) are called *linear combinations of x and y*. More generally, if $a_1x_1 + a_2x_2 + ... + a_nx_n$ are variables and $a_1, a_2, ..., a_n$ are constants, then the expression
$$a_1x_1 + a_2x_2 + ... + a_nx_n$$
is called a *linear combination of the x_i's*. Linear programming is concerned with problems involving such linear combinations.

The general linear programming problem has the following precise mathematical form:
Maximize the linear combination

$$M = c_1x_1 + c_2x_2 + ... + c_nx_n \tag{7}$$

subject to the linear constraints:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \text{ £ } b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \text{ £ } b_2$$
$$... \tag{8}$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \text{ £ } b_m$$
$$x_1 \text{ } ^3 \text{ } 0, x_2 \text{ } ^3 \text{ } 0, ..., x_n \text{ } ^3 \text{ } 0$$

where the c_i's, b_i's and a_{ij}'s are given constants.

Any set of values for the x_i's that satisfies all the inequalities of (8) is called a *feasible solution* of the problem. A set of values for the x_i's that maximizes M is called an *optimal solution*. The linear programming problem asks for an optimal, feasible solution.

We will briefly summarize here the basic results about solutions of linear programming problems. Each inequality of (8) determines a closed half-space of Euclidean n-dimensional space R^n. The intersection of the $M + n$ half-spaces of (8) gives the set of all feasible values for the x_i's. This set, called the *feasibility set* or *constraint set,* has a very special form. It turns out to be what mathematicians call a *polygonal convex set.*

If some of the constraints are mutually inconsistent, then the feasibility set turns out to be empty. In this case, the linear programming problem has no solution. Even if the constraint set is nonempty, the problem may still have no solution; see Exercise 12 for an example. For this to happen, it is necessary, but not sufficient, that the feasibility set be unbounded.

If the linear programming problem has a solution, however, then it always has one that occurs at one of the vertex points of the constraint set. Since there are a finite number of constraints, there will be only a finite number of vertices.

In theory, one could solve any linear programming problem by finding all of the vertices and then computing the value of M at each vertex. In practice, this is not a reasonable way to proceed. Even for a moderate number of constraints, the feasibility set may have a very large number of vertices. It would be a formidable task, even for a computer, to determine all the coordinates of all the vertices.

The algorithm devised by Dantzig and refined by others finds the solution in a much more economical fashion. The strategy behind Dantzig's approach is simple. Start with any feasible solution that is a vertex of the constraint set. Then move to a "nearby" vertex lit which M has a greater value. Repeat this process until you arrive at an optimal solution.

Dantzig's algorithm, called the *simplex method,* not only tells how to move from one vertex to another, it also helps find a feasible solution at which to start and it gives a method for determining when we have arrived at an optimal, feasible solution.

We will devote the major part of this chapter to an elaboration of the ideas of the past few paragraphs. Before we do this, however, we will describe some problems that can be formulated using linear programming models.

D. More Examples of LP Problems

1. *The Breakfast Problem.* Zoey and Sydney will only eat cereal for breakfast. In fact, they will eat only Brand X or Yukkies. Their mother Sherry is concerned about the children receiving adequate nourishment in the morning. According to the box tops of the cereals, one ounce of Brand X provides 1/3 milligram of iron and 3.8 grams of protein, while the same amount of Yukkies offers 1 mg of iron and 2.4 gm of protein. Sherry wants her children to obtain at least 1 mg of iron and 5 gm of protein from their breakfast cereals, but she wants to provide these levels of nutrients at the lowest possible cost. If one ounce of Brand X costs 4 cents while one ounce of Yukkies sells for 6.5 cents, how should she mix the cereals to accomplish her goals?

We will let x represent the number of ounces of Brand X to be served and y the number of ounces of Yukkies. Then Sherry has the following problem:

$$\text{Minimize } M = 4x + 6.5y$$

subject to the constraints:

$$(1/3)\, x + y \geq 1, \text{ (iron)}$$
$$3.8x + 2.4y \geq 5, \text{ (protein)}$$
$$x \geq 0,\ y \geq 0.$$

This type of problem occurs quite frequently in applications. It does not precisely fit the format of a linear programming problem as we defined it, but we can quickly change that. We need only note that an inequality of the form $ax + by \geq c$ is equivalent to the inequality $-ax - by \leq -c$, and that the minimum value of a quantity M is equal to the negative of the maximum value of $-M$. Thus we can write Sherry's problem as:

$$\text{Maximize } \bar{M} = -M = -4x - 6.5y$$

subject to the constraints:

$$-(1/3)\, x - y \leq -1, \text{ (iron)}$$
$$-3.8x - 2.4y \leq -5, \text{ (protein)}$$
$$x \geq 0,\ y \geq 0.$$

so that it fits the definition of an **LP** problem.

For numerical computations, it is sometimes convenient if the coefficients of the constraints are integers. In this problem, we may multiply

the first constraint by 3 and the second by 10 to obtain the equivalent problem:

$$\text{Maximize } M = -4x - 6.5y$$

subject to the constraints:

$$x + 3y \geq 3$$
$$38x + 24y \geq 50 \tag{9}$$
$$x \geq 0, \; y \geq 0$$

2. *A Smuggling Problem.* The Turkish Poppy Company imports heroin into the United States to 20 different dealers through 5 different ports. Let x_{ij} denote the number of pounds to be shipped from port i to dealer j. Since the dealers are located in different parts of the country, there is an associated cost, c_{ij}, of sending 1 pound of heroin from port i to dealer j. The total shipping cost M is given by a double sum:

$$m = \sum_{j=1}^{20} \sum_{i=1}^{5} c_{ij} x_{ij}$$

and the company would like to minimize this cost.

There are two important constraints operating here. First, each dealer has ordered a particular amount of heroin that he believes he can sell in his area of the market. Thus the company must satisfy the order of each dealer. If the *jth* dealer has ordered d_j pounds, then we have a constraint of the form

$$\sum_{i=1}^{5} x_{ij} = d_j$$

We have one such constraint for each of the 20 dealers.

The second type of constraint arises because the heroin must be smuggled into the United States. There are different security measures at each of the five ports of entry, so that different amounts of heroin can be smuggled through different places. Suppose that the smuggling capacity of the *i*th port is s_i pounds of heroin. Then the total number of pounds shipped from each port cannot exceed the smuggling capacity of that port. We formulate this restriction as

$$\sum_{j=1}^{20} x_{ij} \leq s_i, \; i = 1, 2, 3, 4, 5$$

This description of the smuggling problem requires two transformations to convert it into a standard LP problem. First, we need to change the minimizing requirement to a maximizing one; we saw how to do this when we discussed the breakfast problem. Second, we notice that some of the constraints are equations rather than inequalities. This difficulty is also easily remedied. We simply note that the equation

$$ax + by = c$$

is equivalent to insisting that the inequalities
$$ax + by \geq c \quad \text{and} \quad ax + by \leq c$$

both hold. Thus we replace each constraint of the form $\sum_{i=1}^{5} x_{ij} = d_j$ by the pair:

$$\sum_{i=1}^{5} x_{ij} \pounds d_j \quad \text{and} \quad \sum_{i=1}^{5} -x_{ij} \pounds -d_j$$

The smuggling problem is an example of what is called a "transportation problem." Many of the earliest applications of linear programming were to transportation problems and these still provide a fair share of LP work. It is not unusual today to solve transportation problems involving as many as 3,000 constraints and 15,000 variables.

3. *An Assignment Problem.* Mary Muttoni is the chairperson of the history department at a small university. One of her duties is to make up the teaching schedule. The catalog of the university promises that the department will offer four large lecture courses for freshmen next term. These are:

> History A: A Survey of American History,
> History B: Revolutions and Counterrevolutions,
> History C: European Intellectual History,
> History D: China and Japan.

There are four professors in the department who can teach any of the four courses. Because of their different backgrounds, expertise, and enthusiasms, they will attract different numbers of students in each course. Muttoni estimates the student appeal of each instructor in each course and derives a set of enrollment estimates. These are displayed in Table 5.2.

Each professor will be assigned to only one course and each course is to be taught by only one faculty member. The chairperson wishes to maximize the total enrollment in the four courses by assigning the available professors to the different courses.

Table 5.2

Course Professor	A	B	C	D
Doggoff	310	260	270	290

Josephs	270	330	250	210
Reapingwillst	210	230	190	280
Cragdodge	240	210	220	200

We can formulate this problem using linear programming. Let x_{ij} be the variable which is equal to 1 if the ith professor is assigned to the jth course and 0 otherwise. Thus there are 16 variables in the problem. Let e_{ij} be the number of students the ith professor will attract if he teaches the jth course.

The chair's problem is:

$$\text{Maximize } M = \sum_{i=1}^{4}\sum_{j=1}^{4} e_{ij}x_{ij}$$

subject to the constraints:

$$\sum_{i=1}^{4} x_{ij} = 1 \ (j = 1\ 2\ 3\ 4) \qquad \text{(Each course is assigned to one professor.)}$$

$$\sum_{j=1}^{4} x_{ij} = 1 \ (i = 1\ 2\ 3\ 4) \qquad \text{(Each professor is assigned one course.)}$$

and each $x_{ij} \geq 0$.

E. Vector Formulation of the LP Problem

We wish to give a compact statement of the linear programming problem using vector notation. We will denote vectors in this chapter by boldface type.

DEFINITION (See Appendix II.) If A is an $m \times n$ matrix and x is an $n \times 1$ vector, then Ax is the $m \times 1$ vector whose ith component is the product of the ith row of A and the vector x; that is,

$$\mathbf{Ax} = \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{pmatrix}$$

where $y_i = a_{i1}x_1 + a_{i2}x_2 + ... + a_{i}x_n$

DEFINITION If A and B are matrices of the same size, then $A \leq B$ means that $a_{ij} \leq b_{ij}$ for all entries of the two matrices. In particular, if x and y are $k \times 1$ vectors, then x $\leq$ y if and only if

$$x_1 \leq y_1, \quad x_2 \leq y_2, \quad ..., \text{ and } x_k \leq y_k.$$

The following theorem is easily proved and shows that the notion of inequality for matrices has the same features as inequalities for numbers.

THEOREM 1 Suppose $A, B,$ C, and D are matrices of the same size. Then
a) If $A \leq B$ and $B \leq C$, then $A \leq C$;
b) If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$;
c) If $A \leq B$, then $cA \leq cB$ for any positive constant c and $cA \geq cB$ for any negative constant c.

With these properties in mind, we can restate the general linear programming problem:

Find an $n \times 1$ vector **x** such that

$$M = \mathbf{c} \cdot \mathbf{x} \text{ is maximized} \qquad \qquad (7')$$

subject to the constraints

$$\mathbf{Ax} \leq \mathbf{b}, \qquad \qquad (8')$$
$$\mathbf{x} \geq \mathbf{0}.$$

where **c** is a given $1 \times n$ vector, **b** is a given $M \times 1$ vector, A is a given $m \times n$ matrix and **0** is the zero vector, all of whose components are 0.