Introduction to Duality

Fromage Cheese Company Problem	*Dual Problem*
Maximize $4.5x + 4y$ subject to $30x + 12y \le 6000$ (Cheddar) $10x + 8y \le 2600$ (Swiss) $4x + 8y \le 2000$ (Brie) $x, y \ge 0$ x and y are number of packages of each assortment to prepare	Minimize $6000C + 2600S + 2000B$ subject to $30C + 10S + 4B \ge 4.5$ $12C + 8S + 8B \ge 4$ $C, S, B \ge 0$ C,S,B are the price per ounce to offer for the cheeses

Primal Problem	*Dual Problem*
Maximize $Z = \mathbf{cx}$ subject to $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$	Minimize $W = \mathbf{yb}$ subject to $\mathbf{y}A \ge \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$

Alternative 1 for Dual	*Alternative 2 for Dual*
Minimize $W = \mathbf{b}^T \mathbf{y}$ subject to $A^T\mathbf{y} \ge \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$	Maximize $W = -\mathbf{b}^T\mathbf{y}$ subject to $-A^T\mathbf{y} \le \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$

Theorem: The dual of the dual is the primal.

Weak Duality Property: If $\mathbf{x}$ is a feasible solution to the primal problem and $\mathbf{y}$ is a feasible solution of the dual problem, then $\mathbf{cx} \le \mathbf{yb}$.

Corollary 1: Any feasible solution of the dual gives a bound for the primal.

Corollary 2: Any feasible solution of the primal gives a bound for the dual.

Corollary 3: If the primal is unbounded, then the dual is infeasible.

Corollary 4: If primal and dual both have feasible solutions, then both have optimal solutions.

Corollary 5: Suppose $\mathbf{x}$ is feasible for primal and $\mathbf{y}$ is feasible for dual.

If $\mathbf{cx} = \mathbf{yb}$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions.

Strong Duality Property: If $\mathbf{x}^*$ is an optimal solution for the primal primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then $\mathbf{cx}^* = \mathbf{yb}^*$.

Complementary Solutions Property: At each iteration, the simplex method simultaneously identifies a CPF solution **x** for the primal problem and a complementary solution **y** for the dual problem (in objective function row as the coefficients of the slack variables) where **cx = yb**. If **x** is not optimal for the primary problem, then **y** is not feasible for the dual problem.

Complementary Optimal Solutions Property*:* At the final iteration, the simplex method simultaneously identifies an optimal solution **x*** for the primal problem and a complementary optimal solution **y*** for the dual problem where **cx*** = **y*b**. **The components of y** are the shadow prices for the primal problem).

Symmetry Property: For any primal problem and its dual problem, all relationships between them must be symmetric because the dual of this dual problem is the primal problem.

Duality Theorem: The following are the only possible relationships between the primal and dual problems:

(1) If one problem has *feasible solutions* and a *bounded* objective function (and therefore has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.

(2) If one problem has *feasible solutions* and an *unbounded* objective function (and hence *no optimal solution*), then the other problem has *no feasible solutions.*

(3) If one problem has *no feasible solutions*, then the other problem has *no feasible solutions* or an *unbounded* objective function.

Final Tableau of Fromage Cheese Company Problem

	Z		*x*	*y*	*u*	*v*	*w*	
	1		0	0	**0**	**5/12**	**1/12**	1250
x	0		1	0	0	1/6	-1/6	100
y	0		0	1	0	-1/12	5/24	200
u	0		0	0	1	-4	5/2	600