Degeneracy and Cycling in the Simplex Method

A Variation of the Fromage Cheese Company Problem

The Original Problem

Maximize $Z = 4.5x + 4y$
subject to the constraints :

$$30x + 12y \leq \mathbf{6000} \quad (Cheddar)$$
$$10x + 8y \leq \mathbf{2600} \quad (Swiss)$$
$$4x + 8y \leq \mathbf{2000} \quad (Brie)$$
$$x \geq 0, \ y \geq 0$$

Suppose we transposed the supplies of Swiss and Brie and we should have

Maximize $Z = 4.5x + 4y$

subject to the constraints :

$$30x + 12y \leq \mathbf{6000} \quad (Cheddar)$$

$$10x + 8y \leq \mathbf{2000} \quad (Swiss)$$

$$4x + 8y \leq \mathbf{2600} \quad (Brie)$$

$$x \geq 0, \ y \geq 0$$

STEP 1: Introduce slack variables to convert inequalities into equations.

Find nonnegative numbers x, y, u, v, w such that

$$Z = 4.5x + 4y \text{ is maximized}$$

subject to the constraints :

$$\begin{aligned} 30x + 12y + u &= 6000, \\ 10x + 8y + v &= 2000, \\ 4x + 8y + w &= 2600, \end{aligned}$$

$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0,$

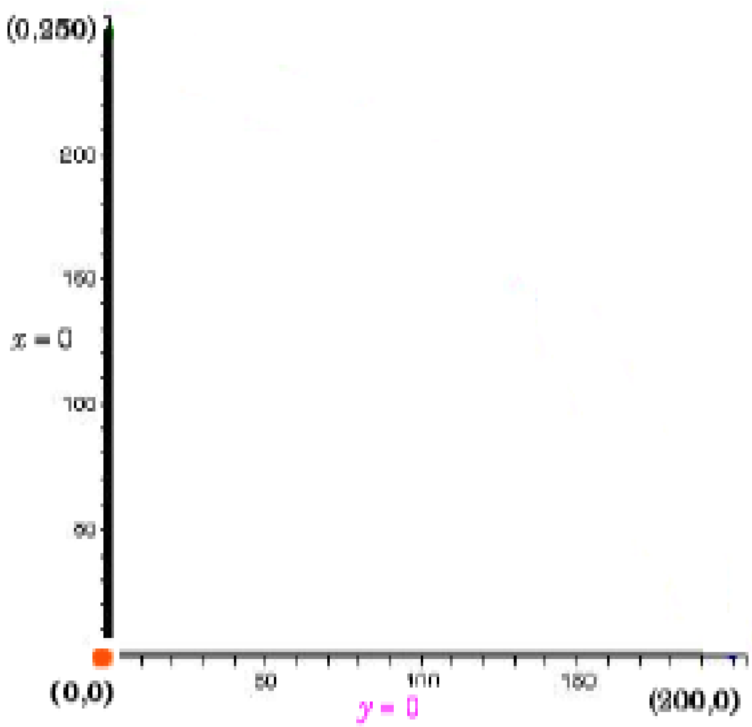
For this cheese example, the solution

$x = 0, \quad y = 0,$
$u = 6000, \quad v = 2000, \ w = 2600.$

is both feasible and basic.

The basic variables are u, v, and w.

Geometrically, this solution is located at the vertex where the two edges $x = 0$ and $y = 0$ intersect.



This particular solution gives $Z = 0$, which is clearly not optimal. We can increase $Z = 4.5x + 4y$ by increasing either x or y.

One way to go about this is to concentrate on increasing one of the variables.

Since a unit increase in x boosts Z more than a unit increase in y, it is reasonable to begin by making x as large as possible, while keeping $y = 0$. When $y = 0$, our equations can be written

$$u = 6000 - 30x,$$
$$v = 2000 - 10x,$$
$$w = 2600 - 4x.$$

Increase x as much as possible until we drive one of the current basic variables to 0.

$u = 0$ when $30x = 6000$; that is, $x = 6000/30 = 200$

$v = 0$ when $10x = 2600$; that is, $x = 2000/10 = 200$

$w = 0$ when $4x = 2000$; that is, $x = 2600/4 = 650$

We'll put the equation for the objective function first:

$$Z - 4.5x - 4y = 0$$
$$30x + 12y + u = 6000,$$
$$10x + 8y + v = 2000,$$
$$4x + 8y + w = 2600.$$

Write the matrix of coefficients in an *extended simplex tableau* (Tableau 1).

Tableau 1

	Z	*x*	*y*	*u*	*v*	*W*	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2000
w	0	4	8	0	0	1	2600
		↑					

x will enter the basis

Tableau 2

		Z	x	y	u	v	w	
	Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
$\frac{6000}{30} = 200$	u	0	[30]	12	1	0	0	6000
$\frac{2000}{10} = 200$	v	0	10	8	0	1	0	2000
$\frac{2600}{4} = 650$	w	0	4	8	0	0	1	2600
			↑					

x will enter the basis
u will leave the basis

We have a tie so either *u* or *v* could have been picked.

Divide *u*-row by 30:

Tableau 3

	Z	X	y	u	v	w	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	1	2/5	1/30	0	0	200
v	0	10	8	0	1	0	2000
w	0	4	8	0	0	1	2600
		↑					

Subtract (-4.5) **u*-row from *Z*-row
Subtract (10) * *u*-row from *v*-row
Subtract (4) * *u*-row from *w*-row

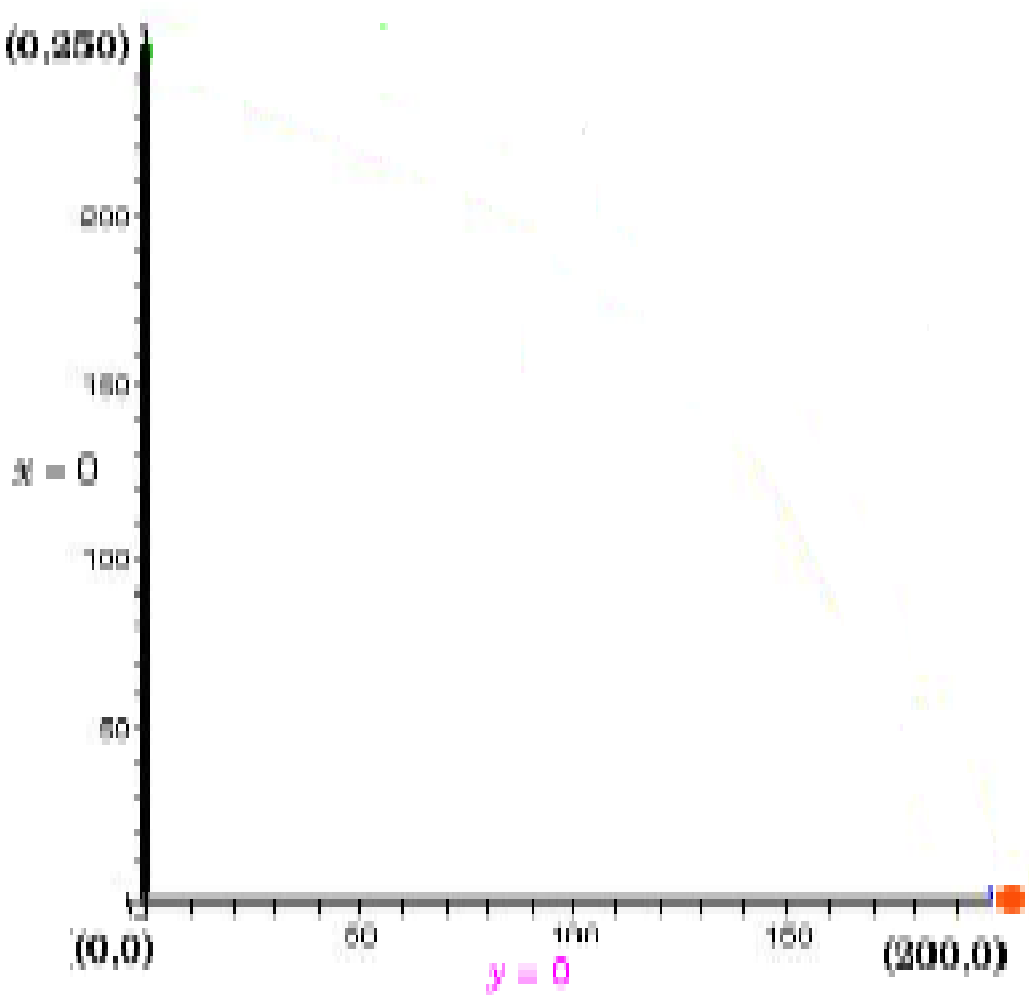
Tableau 4

	Z	x	y	u	v	w	
Z	**1**	**0**	**-11/5**	**3/20**	**0**	**0**	**900**
x	0	1	2/5	1/30	0	0	200
v	**0**	**0**	**4**	**- 1/3**	**1**	**0**	**0**
w	0	0	32/5	- 2/15	0	1	1800

MAJOR OBSERVATION: One of the basic variables has value 0.

Each of the non-basic variables is always 0, but here so is a basic variable.

This situation is known as ***DEGENERACY*.** Degeneracy occurs frequently in Linear Programming Problems solved by the Simplex Method



What's the Big Deal About Degeneracy?

Tableau 5

$\frac{200}{2/5} = 500$

$\frac{0}{4} = 0$

$\frac{1800}{32/5} = 281\frac{1}{4}$

	Z	x	y	u	v	w	
Z	**1**	**0**	**-11/5**	**3/20**	**0**	**0**	**900**
x	0	1	2/5	1/30	0	0	200
v	0	0	**[4]**	- 1/3	1	0	0
w	0	0	32/5	- 2/15	0	1	1800
			↑				

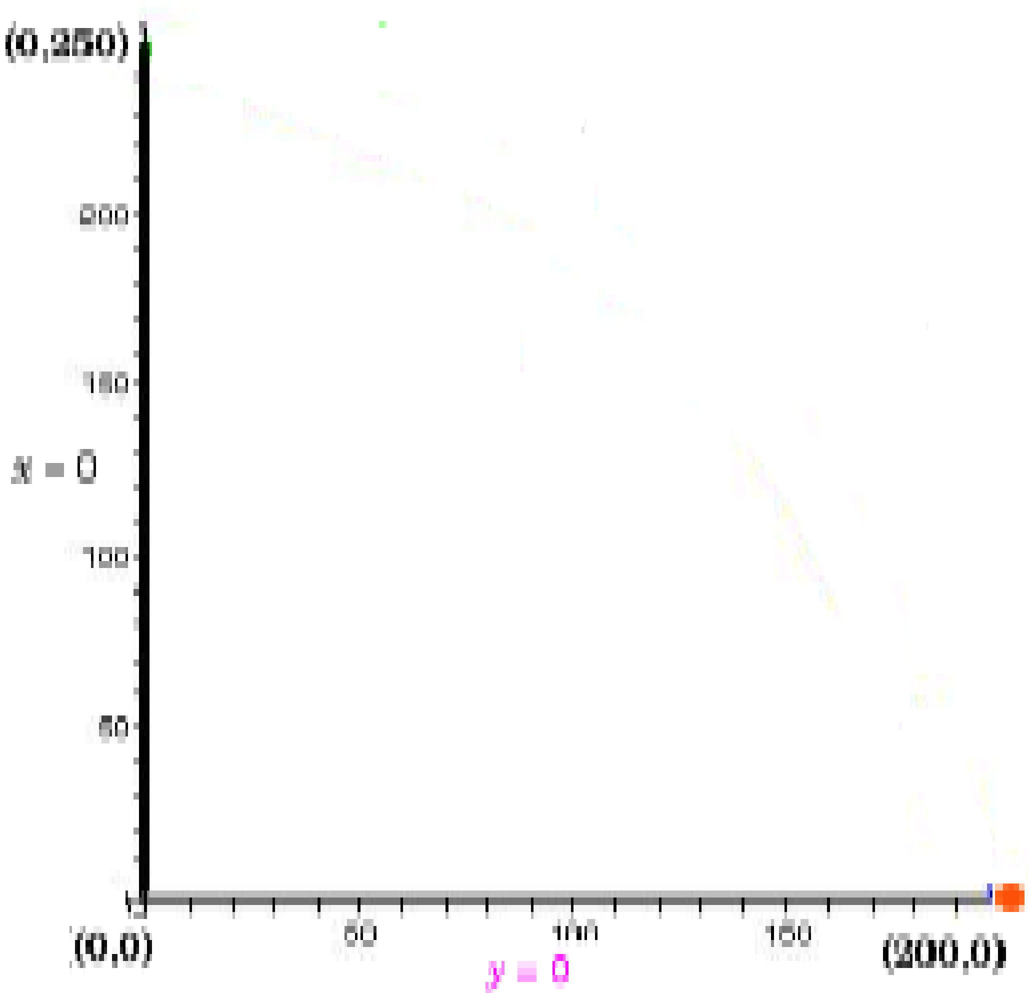
y will enter the basis
v will leave the basis

Divide v-row by 4
Subtract (-11/5)* new v-row from Z-row
Subtract (2/5)* new v-row from x-row
Subtract (32/5)* new v-row from w-row

Result is

Tableau 6

	Z	x	y	u	v	w	
Z	**1**	**0**	**0**	**-1/30**	**11/20**	**0**	**900**
x	0	1	0	1/15	-1/10	0	200
y	0	0	1	- 1/12	1/4	0	0
w	0	0	0	2/5	-8/5	1	1800



We did not move to a new vertex of the constraint set.

Tableau 7

		Z	x	y	u	v	w	
	Z	**1**	**0**	**0**	**-1/30**	**11/20**	**0**	**900**
$\frac{200}{1/15} = 3000$	x	0	1	0	**[1/15]**	-1/10	0	200
	y	0	0	1	- 1/12	1/4	0	0
$\frac{1800}{2/5} = 4500$	w	0	0	0	2/5	-8/5	1	1800
					↑			

u will enter the basis
x will leave the basis

Divide *x*-row by 1/15

Subtract (-1/30) * new *x*-row from *Z*-row

Subtract (-1/12)* new *w*-row from *y*-row

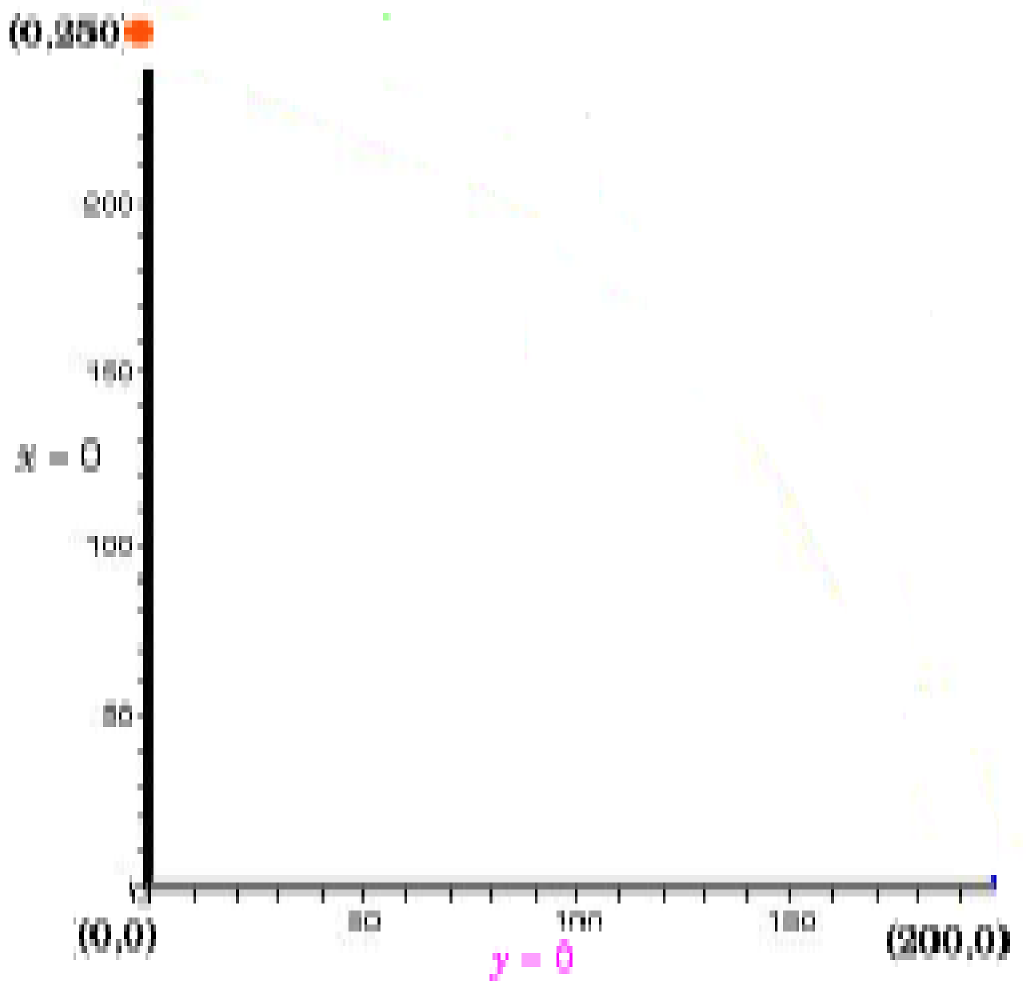
Subtract (2/5) * new *x*-row from *w*-row

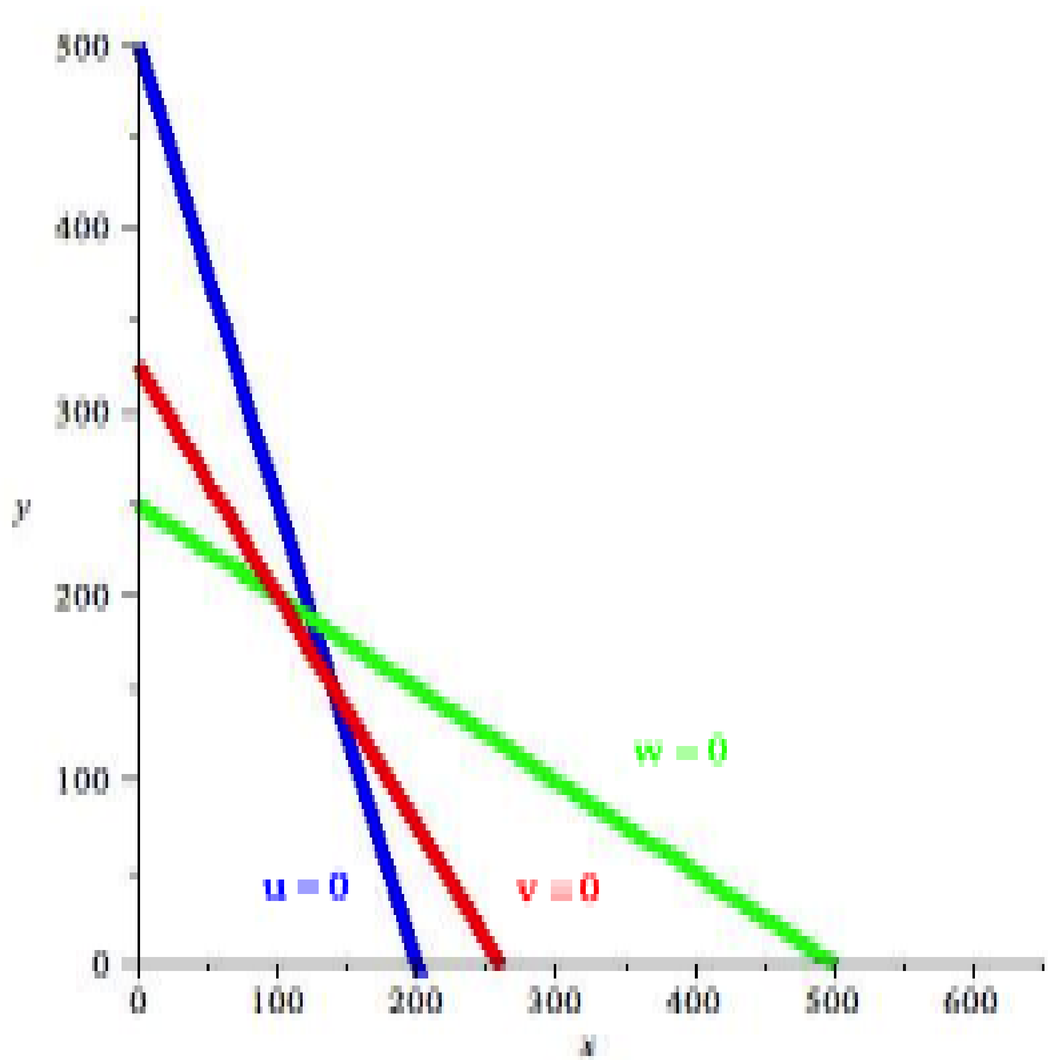
Tableau 8

	Z	x	y	u	v	w	
Z	**1**	**1/2**	**0**	**0**	**1/2**	**0**	**1000**
u	0	15	0	1	-3/2	0	3000
y	0	5/4	1	0	1/8	0	250
w	0	-6	0	0	-1	1	600

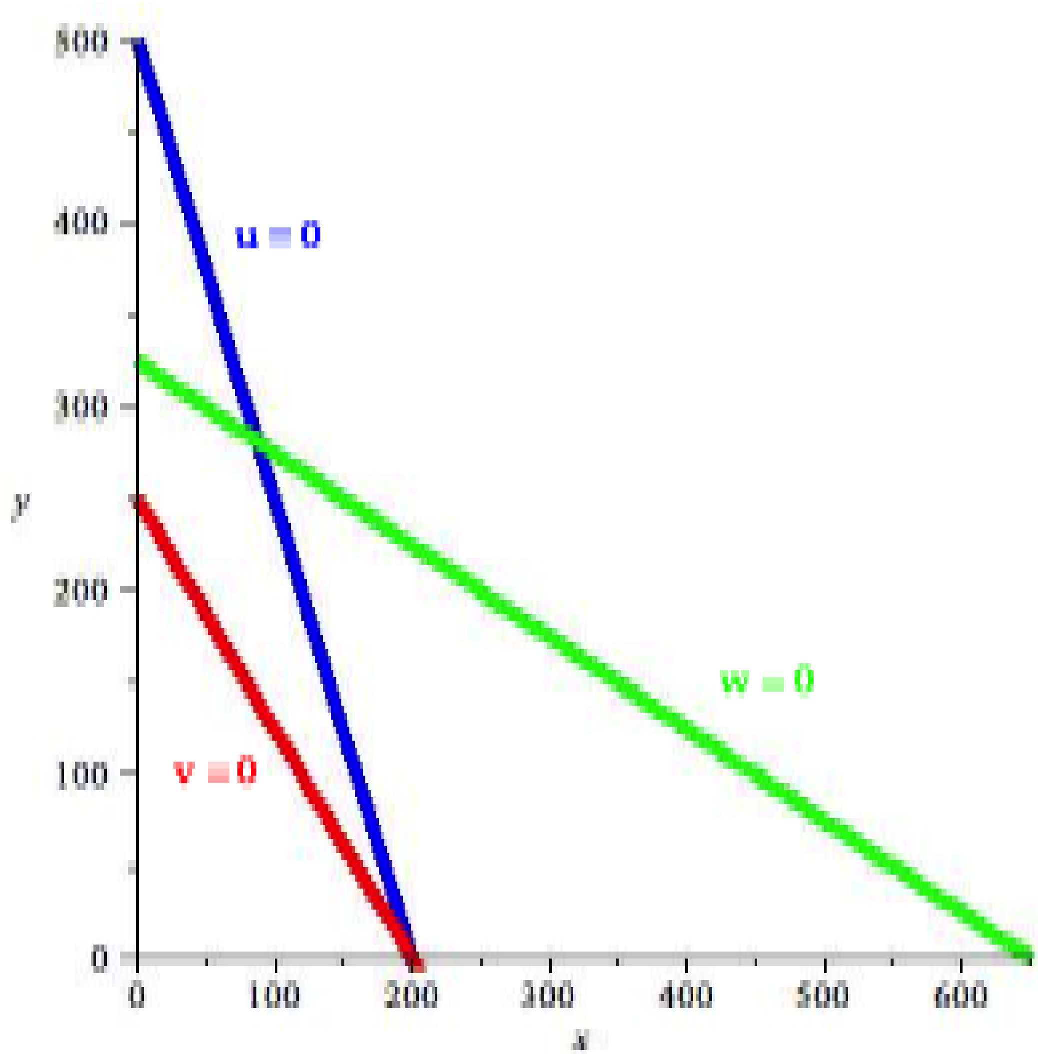
The new basic feasible and optimal solution is

w= 600, y = 250, u = 3000, x = v = 0, and Z = 1000.





Original Fromage Problem



New Fromage Problem

CYCLING

When Degeneracy occurs, some basic variable has value 0.

If that variable leaves the basis at the next iteration, the value of the objective function and the basic variables will not change.

Cycling is Possible Under the Simplex Method

See Last Problem in Assignment 4

Avoiding Cycling in the Simplex Method

Bland's Rule: Use the following rules to determine the entering and leaving variables for a simplex pivot:

- Among all of the variables eligible to *enter* the basis, choose the one with **smallest index**.

- Among all of the variables eligible to *leave* the basis, choose the one with **smallest index**.

Theorem: If Bland's Rule is used to choose the entering and leaving variables in the simplex method, then the simplex method will never cycle.

Robert G. Bland, "New Finite Pivoting Rules For The Simplex Method," *Mathematics of Operations Research* **2** (1977), 103 – 107.

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NEW FINITE PIVOTING RULES FOR THE SIMPLEX METHOD*†

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A simple proof of finiteness is given for the simplex method under an easily described pivoting rule. A second new finite version of the simplex method is also presented.

1. A simple finite pivoting rule. Consider the canonical linear programming problem

$$\begin{aligned} &\text{maximize} \quad x_0, \\ &\text{subject to} \quad Ax = b, \\ &\qquad x_j \geqslant 0 \quad \forall j \in E = \{1, \ldots, n\}, \end{aligned} \tag{1.1}$$

where A has $m + 1$ rows and $n + 1$ columns and is of full row rank. We denote the canonical simplex tableau for (1.1) corresponding to some basic set of variables with index set $B = \{B_0 = 0, B_1, \ldots, B_m\}$ by $(\bar{A}, \bar{b})$. It is assumed that the rows of $(\bar{A}, \bar{b})$ are ordered so that $\bar{a}_{i, B_i} = 1$; thus the ith row of the tableau represents the equation $x_{B_i} + \sum_{j \notin B} \bar{a}_{ij} x_j = \bar{b}_i$. If $\bar{b}_i \geqslant 0$ for $i = 1, \ldots, m$, then the tableau is (primal) feasible and the simplex pivoting rule permits the selection of any (nonbasic) variable x_k having $\bar{a}_{0k} < 0$ to enter the basis. If $\bar{a}_{0j} \geqslant 0$ for all $j \in E$, then the pivoting stops with the current tableau optimal. Having chosen a variable x_k to enter the basis, the simplex rule permits the selection of any basic variable x_{B_r} having $\bar{a}_{rk} > 0$ and

$$\frac{\bar{b}_r}{\bar{a}_{rk}} = \min\left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} : \bar{a}_{ik} > 0 \right\}$$

to leave the basis. If $\bar{a}_{ik} \leqslant 0$ for $i = 1, \ldots, m$, then the pivoting stops with the current tableau indicating primal unboundedness and dual infeasibility.

A pivoting rule that is consistent with the simplex rule and further restricts the choice of either the pivot column or the pivot row is called a *refinement* of the simplex rule. We say that a refinement determines *a simplex* method, as opposed to *the simplex method*, which is used here as a generic term referring to the family of methods determined by all possible refinements.

It is very well known that the simplex method can fail to be finite because of the possibility of cycling. Certain refinements of the simplex pivoting rule, such as the lexicographic rule described in [3], restrict the selection of the pivot row in such a way that cycling cannot occur. The following refinement, which restricts the choice of both the pivot column and the pivot row, determines a simplex method that is, among all finite simplex methods known to us, the easiest to state, the easiest to implement, and the easiest to prove finite.

Let Rule I be the refinement of the simplex pivoting rule obtained by imposing the following restriction:

among all candidates to enter the basis, select the variable x_k having the lowest

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Last Problem of Assignment 4: Find where Bland's Rule is violated.

Extra Credit: Prove that Bland' s Rule prevents cycling.

Theorem: If cycling does not occur, then the Simplex Method will find the extreme vale of the objective function.

Proof: There are only finitely many sets of basic variables.

Note: The number of iterations could be quite large.

Suppose we have $n = 60$ original decision variables and $m = 50 \leq$ constraints.

Then we introduce 50 new slack variables. The number of possible bases could be on the order of

$$\binom{110}{50} = \frac{110!}{50!60!} \sim 6.28\, 10^{31}$$

Some Linear Algebra Behind the Simplex Method

Original Problem: n decision variables, m constraints

Fromage: $n = 2$, $m = 3$
Chairs: $n = 3$, $m = 3$

Augment with m slack variables so we can represent constraint set as the solution set of a system of linear equations with $(n + m)$ variables and m equations.

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

where we can write **A as**

$$\mathbf{A} = (\mathbf{B}, \mathbf{N})$$

where **B** is an ***m*** by ***m*** invertible matrix and

$$\vec{\mathbf{x}} = \begin{pmatrix} \overrightarrow{\mathbf{x}_{\mathbf{B}}} \\ \overrightarrow{\mathbf{x}_{\mathbf{N}}} \end{pmatrix}$$

For Original Fromage:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 30 & 12 \\ 0 & 1 & 0 & 10 & 8 \\ 0 & 0 & 1 & 4 & 8 \end{pmatrix} \quad \text{(columns } u \; v \; w \; x \; y\text{)}$$

and

$$\vec{\mathbf{x}} = \begin{pmatrix} u \\ v \\ w \\ x \\ y \end{pmatrix}$$

Then we can write

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

as

$$(\mathbf{B},\mathbf{N}) \begin{pmatrix} \overrightarrow{\mathbf{x}_{\mathbf{B}}} \\ \overrightarrow{\mathbf{x}_{\mathbf{N}}} \end{pmatrix} = \vec{\mathbf{b}}$$

$$\mathbf{B}\overrightarrow{\mathbf{x}_{\mathbf{B}}} + \mathbf{N}\overrightarrow{\mathbf{x}_{\mathbf{N}}} = \vec{\mathbf{b}}$$

$$\mathbf{B}\overrightarrow{\mathbf{x}_{\mathbf{B}}} = \vec{\mathbf{b}} - \mathbf{N}\overrightarrow{\mathbf{x}_{\mathbf{N}}}$$

$$\overrightarrow{\mathbf{x}_{\mathbf{B}}} = \mathbf{B}^{-1}\ \vec{\mathbf{b}} - \mathbf{B}^{-1}\mathbf{N}\overrightarrow{\mathbf{x}_{\mathbf{N}}}$$

A *basic* solution is one in which

$$\overrightarrow{\mathbf{x}_{\mathbf{N}}} = \vec{\mathbf{0}}$$

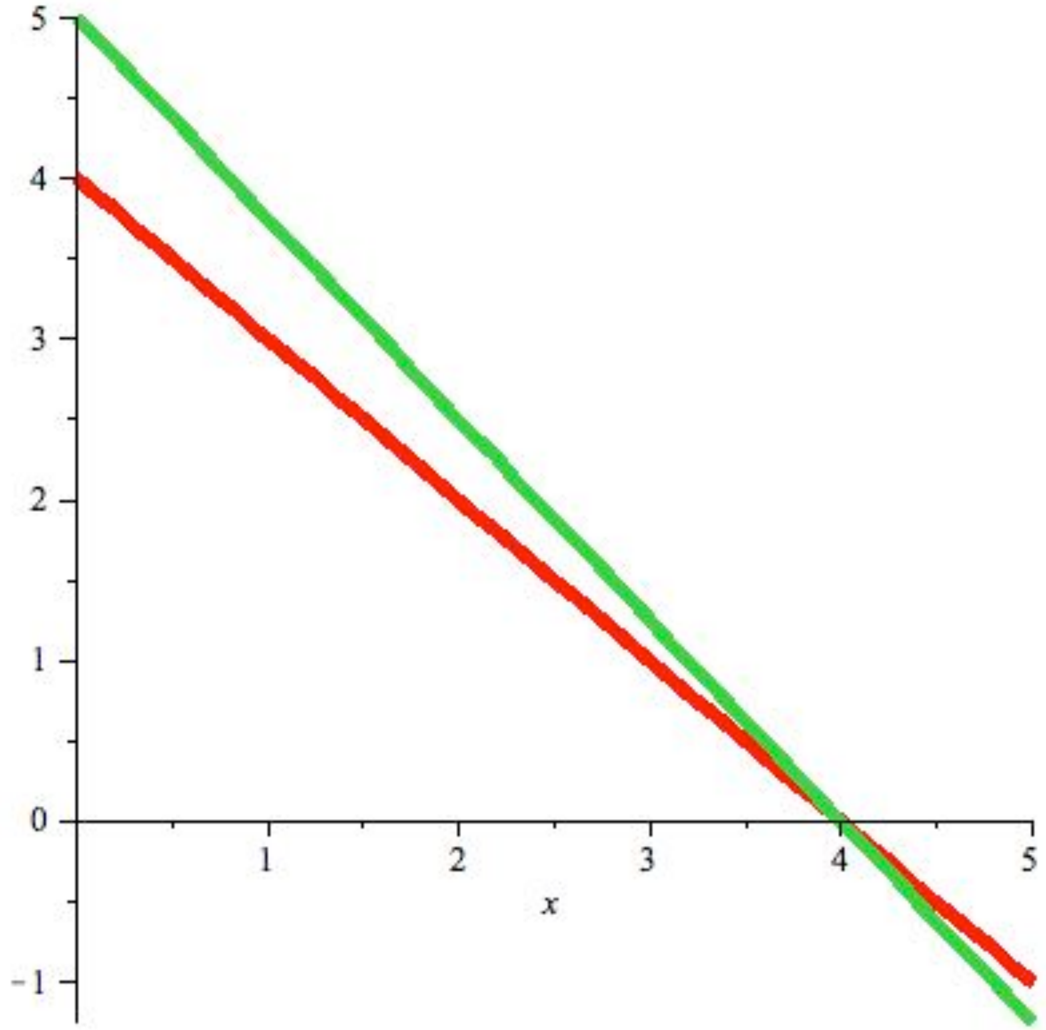
A *basic feasible solution* is a basic solution if

$$\mathbf{B}^{-1}\ \vec{\mathbf{b}} \geq \vec{\mathbf{0}}$$

The calculations are easy if **B** is the identity matrix.

Example. Suppose the constraint set is given by

$$x + y \leq 4$$
$$5x + 4y \leq 20$$
$$x \geq 0,\ y \geq 0$$



Convert to equations

$$\begin{aligned} 1x + 1y + 1u \quad &= 4 \\ 5x + 4y \quad + 1v &= 20 \end{aligned}$$
$$x, y, u, v \text{ all} \geq 0$$

$$\begin{matrix} & x & y & u & v \\ \mathbf{A} = & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{pmatrix} \end{matrix}$$ and $$\vec{\mathbf{b}} = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

Let $$\begin{matrix} & x & y \\ \mathbf{B} = & \begin{pmatrix} 1 & 1 \\ 5 & 4 \end{pmatrix} \end{matrix}$$

Then

$$\mathbf{B}^{-1}\vec{\mathbf{b}} = \begin{pmatrix} -4 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 20 \end{pmatrix} = \begin{pmatrix} -16+20 \\ 20-20 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

so the basic feasible solution is $x = 4$, $y = 4$, $u = 0$, $v = 0$.

But we also could have chosen

$$\begin{matrix} & x & v \\ \hat{\mathbf{B}} = & \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \end{matrix}$$

where y, u are the nonbasic variables. Here

$$\hat{\mathbf{B}}^{-1}\vec{\mathbf{b}} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 20 \end{pmatrix} = \begin{pmatrix} 4+0 \\ -20+20 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

The extreme points corresponding to $\mathbf{B}$ and $\hat{\mathbf{B}}$ might be identical.

Sherry's Breakfast Problem

Minimize $4x + 6.5y$

subject to

$$\begin{aligned} 1)\;\; 1x + 3y &\geq 3 \qquad \text{(iron)} \\ 2)\;\; 38x + 34y &\geq 50 \qquad \text{(protein)} \end{aligned}$$

and $x \geq 0, y \geq 0.$

Step 1. Convert to Maximization Problem

Maximize $Z = -4x - 6.5y$

Step 2: Subtract *surplus* variables from each constraint:

Maximize $Z = -4x - 6.5y$

subject to

$$\begin{aligned} 1)\;\; 1x + 3y - u &= 3 \\ 2)\;\; 38x + 34y - v &= 50 \end{aligned}$$

and $x \geq 0, y \geq 0, u \geq 0, v \geq 0$

Step 3: Add artificial variables to each constraint to generate a basic feasible solution

$$\text{Maximize } Z = -4x - 6.5y$$

subject to

$$\begin{aligned} &1)\ 1x + 3y - u + a = 3 \\ &2)\ 38x + 34y - v + b = 50 \end{aligned}$$

and $x \geq 0, y \geq 0, u \geq 0, v \geq 0, a \geq 0, b \geq 0$

Step 4. Adjust the objective function to make use of artificial variables prohibitively expensive

$$\text{Maximize } Z = -4x - 6.5y - pa - pb$$

OR

$$\text{Maximize } Z = -4x - 6.5y - Ma - Mb$$

Where p (or M) is an unspecified by very large positive number, the penalty for using one of these artificial variables.