INTRODUCTION TO

OPERATIONS RESEARCH MODELS

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THE NATURE OF OPERATIONS RESEARCH

CHAPTER 1

1.1 INTRODUCTION

The term *operations research* (or *operational research* in England) was first used during the early years of World War II. It referred to studies being conducted for the British military command by teams of scientists, engineers and mathematicians. These studies generally involved attempts to deal with tactical and strategic problems associated with military *operations,* thus the origin of the name.

Today operations research refers to the application of the scientific method, generally by interdisciplinary teams of specialists, to any problem relating to the optimal management of some system, or phase of operations of some system, however small or large. The "system" can be military, industrial, governmental, urban, educational, and so on. The important point is that some phase of an *operating system* is being examined with an eye to describing or optimizing its performance.

It is somewhat surprising that before 1940 a discipline with these objectives had never existed in the sense of possessing a well-defined history, set of techniques and practitioners. As the Industrial Revolution proceeded from the eighteenth century through the early part of the twentieth century, the demand for *applied science* fostered the birth and development of many new disciplines. These included mechanical, electrical, chemical and industrial engineering, all of which involved the application of the scientific method, as well as the principles of physics and chemistry, to new problem areas. They also involved the use of some standard mathematical techniques. In addition, special applied scientific disciplines such as applied optics, metallurgy, and ceramic engineering arose to deal with the multiplicity of thorny technical problems that attended the increasing industrialization of society. Industrial engineering did begin to consider certain areas of human and machine performance, but never to the extent exhibited in the subsequent development of operations research. In short, the *technical* problems of a developing industrialization led to the creation of special disciplines to deal with these problems. However, the attendant *management* problems of optimally operating an increasingly complicated set of systems, with sometimes contradictory objectives, did not lead to the development of a discipline of comparable depth and power prior to World War II.

The British success in dealing, for example, with the deployment of radar as a tactical and strategic component of the air defense of Great Britain, led to the use of a similar scientific approach in the United States, where the military commanders were struggling with problems of a kind and magnitude never before encountered in warfare. These involved the operational use of new equipment such as fast planes, sea mines, varied electronic equipment of all kinds and, with it all, very severe and complex problems of logistics, supply and allocation of resources.

The military uses of operations research have continued and multiplied. At the end of the war, throughout the world and particularly in the United States and western Europe, the larger industrial companies also began to form operations research groups. The growth has been rapid. The Operations Research Society of America was founded in 1953 and the International Federation of Operational Research Societies was established in 1959. There are 27 member societies in 25 countries of the world. Universities throughout the world now have programs and departments for the education of students at all levels, and for the pursuit of new areas of research.

1.2 THE NEED FOR OPERATIONS RESEARCH

Before considering the specific nature of operations research and how operations research studies are conducted, let us examine the origin of the complex management and operational problems which operations research seeks to solve.

Consider a large company that manufactures and sells approximately one hundred different products. The executives of the company see their objective as the maximization of profit. For reasons of "efficiency" they have organized the company along certain functional lines. These include purchasing, personnel, finance, accounting, production, marketing and sales, and research and development. Each of these departments has a director who has an "objective" for his department, which is more than likely imposed by the company's executive committee and president. For example, the purchasing department's objective will be to minimize the prices it pays for raw materials. One way to do this, if it is feasible, is to minimize the cost of each material it purchases. However, this is rarely completely possible. Some materials have to be purchased in certain combinations and also in certain fixed quantities. Hence, an optimization problem arises here as to what particular combination of supplies, quantities and materials minimizes the overall cost of purchasing. We see here the beginning of complex interactions among the variables of the system.

Often a company can purchase materials at reduced costs if large quantities are purchased in one order. This in turn may require the manufacturing or production departments to provide storage space for the material for a period of time prior to its use. This involves additional cost. Thus, we see another source of interaction or even conflict of considerations as to what may be "optimal." There are others. In order to increase sales, the marketing people desire as wide a range of products, related or unrelated, as possible. This leads them to urge widening of the product line. Often they will come into conflict with production personnel over products that have low volume and profit or products that are unprofitable and should be dropped from the product line.

There are other pressures and concerns. The personnel department and also the manufacturing departments do not want to hire and lay off production personnel as sales fluctuate. This is disruptive to the production process, involves serious problems of morale and increases costs because hiring, training and termination all involve additional costs. However, the company's financial watchdogs will be urging them to do this because of the impact it will have on the *overall* profitability of the company. The accounting department will also come into conflict with the purchasing and marketing people in their desire to keep inventories of raw materials and finished products at the lowest possible level to sustain sales. A complete list of the possible interactions of this type might fill this entire book.

Two facts can be deduced from the foregoing discussion:

1. Each department, division or other subunit has one or more optimal operating decisions to make in order to optimize the performance of that particular subunit.

2. The overall corporate management has one or more optimal strategic decisions to make regarding the *overall* performance and level of operation, as well as the amount and kind of interaction, of the several operating and/or functional subdivisions of the company.

It should be noted that not all decisions of the several subunits of the company will necessarily be in conflict. Often they will not be. However, it is also frequently the case that they are. In any event, the two kinds of operating decisions alluded to above are the principal origins of the need for a methodology comprising analysis, formulation, solution and application to problems concerning the operation of any complex system, whether it be industrial, military, social or governmental.

In order to make the preceding discussion less abstract, let us consider the following simple example of the "balancing" of possibly competing objectives that is present in all attempts to optimize a system.

EXAMPLE 1

A manufacturer of gears has to supply a customer who incorporates these gears into his manufacturing and assembly operation. The customer has a fixed definite order for 50,000 gears per year. Because the customer's order is large, there is ample competition for his business and since the customer is close to the manufacturer, the customer has provided himself with virtually no storage space for the gears. Hence, the gear manufacturer must ship a supply of gears each day to his customer. Because of this the manufacturer faces the following problem. He cannot afford to have a shortage because if he does he may lose the customer's large standing order. On the other hand, he needs his equipment to make other gears for other customers, so he cannot afford to be making production runs too often. There is a cost associated with setting up his equipment between different kinds of production runs. However, if he does not make production runs often, he must store his customer's gears so as to be able to supply him daily. He asks himself the following question: Knowing what my various costs are, how often should I make a production run to satisfy my customer's demands and *minimize* my total cost of operation, i.e., the sum of the costs associated with setting up my machinery and also storing the gears for this customer's supply?

The manufacturer's accountants have told him that the cost of labor and supplies for setting up a production run is \$500 per run. They also state that the cost of keeping a gear in inventory is approximately 15 cents a month. Let us summarize our data. Let

D = demand = 50,000 gears
T = total time of demand = 12 months
C_s = set-up cost = \$500.00
C_I = inventory cost = \$0.15 per gear per month
y = number of gears produced per run (to be determined)

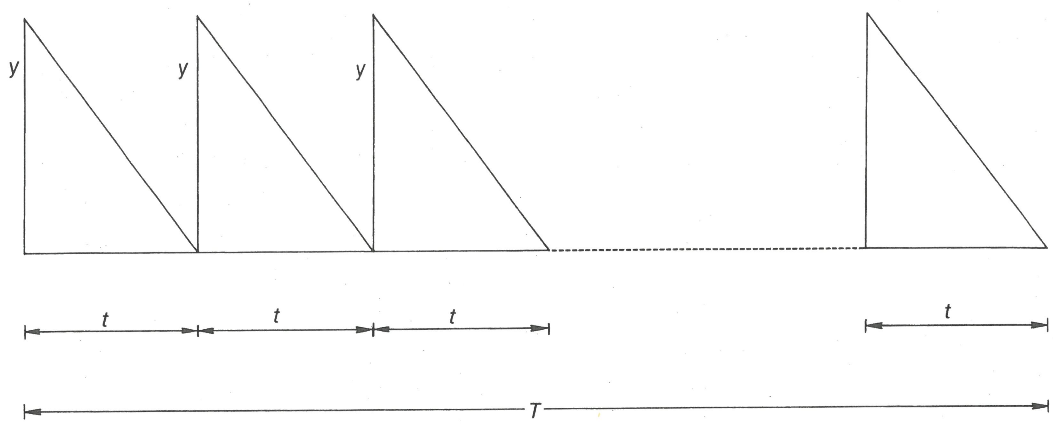

Figure 1–1 The gear manufacturer's problem.

t = time between production runs (to be determined)

S_c = total cost per year (to be determined)

In analyzing this problem the simple diagram shown in Figure 1–1 is quite useful. At the beginning of $T = 12$ months, we must make a run of y gears and t months later another run of y gears, and so on. What should t and y be? As Figure 1–1 shows, the gears are depleted at a uniform rate, hence, the diagonal straight lines. At the time the gears are used up we make another run. Note the unspoken assumption here—the time to make a production run is very small compared to t, the time between production runs, and so can be neglected. If this was not so, we would have to modify the analysis we are about to make.

Since D is the total demand over the period T and y is the number of gears produced per run, then

$$\frac{D}{y} = \text{number of runs during time } T \tag{1}$$

Since T is the total time, then $\frac{T}{D/y}$ will be the time between runs, i.e.,

$$t = \frac{T}{D/y} = \frac{Ty}{D} \tag{2}$$

From Figure 1–1 we see that we begin any period t with y gears and we end it with none. Therefore, the average number of gears in stock during t is $\frac{y}{2}$. Hence, the inventory cost during t is given by:

$$\frac{y}{2} C_I t \tag{3}$$

In addition, we incur a set-up cost each time we make a production run. Hence, our *total cost for each run* is:

$$\frac{y}{2} C_I t + C_s \tag{4}$$

However, we have already noted that the total number of runs during T is, according to equation (1), $\frac{D}{y}$. Therefore, combining equations (1) and (4), we see that the total cost over time T will be:

$$S_c = \left(\frac{y}{2} C_I t + C_s\right) \frac{D}{y} \tag{5}$$

From equation (2) we can substitute for t into (5) to obtain:

$$S_c = \left(\frac{y}{2} C_I \frac{Ty}{D} + C_s\right) \frac{D}{y}$$

which simplifies to:

$$S_c = \frac{C_I T y}{2} + \frac{C_s D}{y} \tag{6}$$

Equation (6) is an expression for the total cost (S_c) of setting up production runs and holding gears in inventory for one year. We need to determine the specific value of y, the number of gears per run, that will make S_c as small as possible.

If we examine equation (6) graphically, we can see clearly why we need to "balance" inventory costs (the first term) against set-up costs (the second term). Figure 1–2 shows these costs separately and combined.

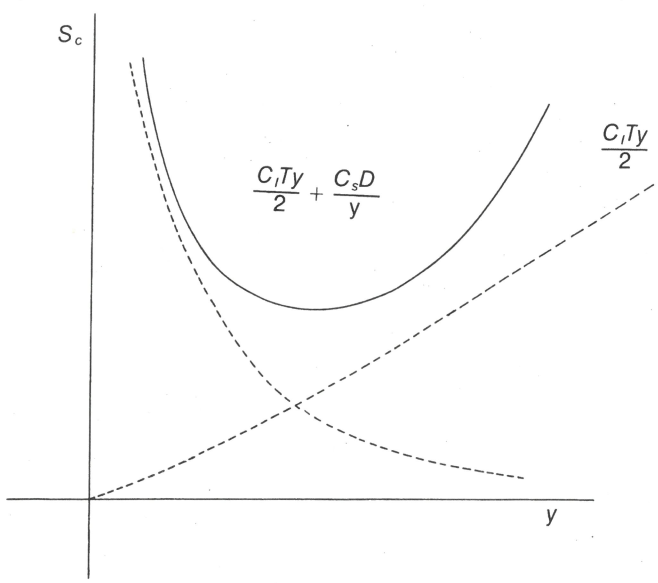

Figure 1–2 Total cost curve for the gear problem.

The larger y is, the greater is the total inventory cost; conversely, the larger y is, the smaller is the total set-up cost. Hence, there is some point, which we seek, that minimizes the *total cost* of both set-up and inventory charges.

Let us now consider how to find the minimum value of y. The equation for total cost is equation (6)

$$S_c = \frac{C_I T y}{2} + \frac{C_s D}{y} \tag{6}$$

We can make use of elementary calculus to find the minimum value of y. If we differentiate S_c with respect to y we know that a minimum must satisfy the relationship:

$$\frac{dS_c}{dy} = 0 \tag{7}$$

Let us then proceed to solve (7) for y_m, the minimum value.

$$\frac{dS_c}{dy} = \frac{C_I T}{2} - \frac{C_s D}{y^2} = 0 \tag{8}$$

If we solve (8) for y_m*, we have:

$$y_m = \left(\frac{2C_s D}{C_I T}\right)^{1/2} \tag{9}$$

From equation (2) we can find t_m, the time between the set-up of production runs. This is:

$$t_m = \frac{T y_m}{D} = \frac{T}{D}\left(\frac{2C_s D}{C_I T}\right)^{1/2} = \left(\frac{2C_s T}{C_I D}\right)^{1/2} \tag{10}$$

The value of S_c corresponding to y_m is given by S_{cm} as:

$$S_{cm} = \frac{C_I T y_m}{2} + \frac{C_s D}{y_m}$$

$$= \left(\frac{C_I T}{2}\right)\left(\frac{2C_s D}{C_I T}\right)^{1/2} + \frac{C_s D}{\left(\frac{2C_s D}{C_I T}\right)^{1/2}}$$

*We know from calculus that for a minimum (as distinct from a maximum) at $y = y_m$, $\frac{d^2S_c}{dy^2}$ must be positive, i.e., $\left.\frac{d^2S_c}{dy^2}\right|_{y=y_m} > 0$. We can see that this is the case, since $\left.\frac{d^2S_c}{dy^2}\right|_{y=m} = \frac{2C_s D}{y_m^3} > 0$ for all values of $y > 0$. Therefore, (9) gives the *minimum* value of y.

Simplifying the preceding equation, we obtain:

$$S_{cm} = (2C_I C_s DT)^{1/2} \tag{11}$$

Let us now return to solving the gear manufacturer's problem. Since we know that $D = 50{,}000$ gears, $T = 12$ months, $C_s = \$500$ per run and $C_I = \$0.15$ per gear per month, we can substitute into equation (9) to obtain:

$$y_m = \left(\frac{2(500)(50{,}000)}{(0.15)(12)}\right)^{1/2} = 5270 \text{ gears per production run}$$

$$t_m = \left(\frac{2(500)(12)}{(0.15)(50{,}000)}\right)^{1/2} = 1.26 \text{ months between runs}$$

$$S_{cm} = [2(0.15)(500)(50{,}000)(12)]^{1/2} = \$9487 \text{ total cost for the year}$$

Any deviation from y_m and t_m in either direction will result in increased yearly costs. In short, assuming the data are accurate, we have found the precise "balance" between inventory charges and set-up costs to minimize the total cost. In subsequent sections we will discuss how, in practice, one should verify the solution to a computation of this kind.

1.3 THE SCIENTIFIC METHOD, MODELS AND OPERATIONS RESEARCH

As we have noted previously, operations research is concerned with applying the scientific method to the optimal management or operation of some relatively complex system. In the physical sciences a common first step in the study of a problem is the use of a variety of experimental techniques to study the range, effect and degree of importance of known variables and often the discovery of the existence and/or importance of hitherto unsuspected variables. Initial exploratory efforts of this kind are of the utmost importance in the experimental sciences. They provide a base of demonstrable fact upon which to build and test an array of hypotheses and theories, that in turn may suggest further experimental tests and data to collect. In the study of industrial, military, governmental or societal system operations, however, it is often not possible, or at least not practical, to engage in experimentation. For example, a company would not wish to risk losing valuable customers in order to conduct an experiment relating to competitive market strategies. However, the need for "experiments" of some sort persists and it is dealt with in operations research as it is often dealt with in engineering and other sciences, when certain kinds of physical experimentation are not convenient. What one can do in such circumstances is formulate and study a *model* of the system of interest.

A model is a *substitute representation* of some object or system in which changes in the components of the model, whatever they may be, affect it in such a way that the results can be translated into the same kind of response in the original object or system under study. Often a model may be *physical.* For example, a structural engineer may study a scaled down (in size) version of a large structural section of a space vehicle. Aircraft design for many years has been strongly influenced by studies of models in wind tunnels. However, the most frequent kind of model employed by engineers and scientists is *nonphysical.* These models are usually called *mathematical models,* because the substitute representation is a mathematical or similar symbolic (e.g., logical) form. The same kinds of mathematical models are employed in an operations research study. For example, an engineer studying the operation of a heat exchanger may wonder how the rate of heat flow will change if he changes the construction material of the shell, or the flow rate of the cooling water through the heat exchanger. Rather than *physically* build a new heat exchanger or actually change the flow rate of the cooling water, he merely changes the values of some numbers in a well known set of mathematical equations used in heat exchanger design. These equations are a *model* of the actual heat exchanger, and changing the numbers in the model should produce the same outcome in the calculated value of heat flow as would be measured by actually making the changes physically on the heat exchanger.

Consider now the example of the gear manufacturer's problem of the previous section. Equation (6) is a mathematical model of how the total cost of storage and set-up of production runs will depend upon the amount that is produced in each run, y. Similarly, equation (9) is a mathematical model of how the optimal amount to be produced, y_m, depends upon the costs, the demand and the time period under consideration.

In brief, these mathematical models, if they are accurate, enable the gear manufacturer, in a few minutes calculation, to determine what otherwise might require very costly and disastrously lengthy experimentation on his manufacturing operations. In practice, he could not afford to undertake it, demonstrating the very great necessity for resorting to mathematical models for important information concerning optimal operating policies.

The general form of a typical model in operations research is as follows:

Maximize (or minimize)

$$P = f(x_i, y_j)$$

subject to: $g_k(x_i, y_j)\ \{\leq, =, \geq\}\ 0 \quad k = 1, 2, \ldots, K$ (12)

where: P = measure of performance of the system
x_i = variables that can be or are being controlled
y_j = constants and variables that cannot be or are not being controlled
f = the relationship between the measure of performance and x_i and y_j
g_k = the relationships ($k \geq 1$) between the controlled variables which tell us between what limits they can be manipulated

In order to comprehend the form of equations (12) more readily, consider equation (6) in the example of the gear manufacturer's problem. It states that

$$S_c = \frac{C_I T y}{2} + \frac{C_s D}{y}$$

The measure of performance (P) is here S_c, the total cost of setting up production runs and holding gears in storage. We wish to minimize S_c. Our controllable variable (x_i) is y, the amount we manufacture in each production run. Our uncontrollable, or fixed or constant variables (y_j) are here C_I, C_s, D, T. This is a straightforward example of the performance relationship of (12). The gear manufacturer had no constraints on the value y. Thus, there were no relationships corresponding to the constraints $g_k(x_i, y_j)\ \{\leq, =, \geq\}\ 0$. (For each of these constraints, if there is more than one, only one of the relationships $\{\leq, =, \geq\}$ can hold; i.e., each of these is either an inequality in one direction or the other, or is an equation.) However, it would be quite possible (and indeed is almost always the case) that there would be a constraint on y of some kind. For example, if the gear manufacturer's supply of raw material or working capital (cash on hand) could not exceed a certain limit, then this *might* limit the allowable size of the production run. We will see examples of this phenomenon in subsequent chapters.

In general, then, an operations research model is a set of equations and/or inequalities among the controllable or manipulatable variables that must be satisfied and a relationship of some performance measure (cost, profit, value) to these controllable variables, as well as other variables and costs. We wish to decide what values the controllable variables should have in order to optimize the performance measure.

There is another kind of model known as a *descriptive* model, which is developed to study relationships between variables and which we assume does not necessarily contain controllable variables. It has important uses in model construction. This is discussed in further detail in Section 1.4.

1.4 THE METHODOLOGY OF OPERATIONS RESEARCH

It is often stated that there are five stages or phases to a properly conducted operations research study. These are:

1. Formulation of the problem.
2. Construction of the model.
3. Solution of the model.
4. Testing of the model and evaluation of the solution.
5. Implementation of the solution.

It is rare that each of these phases can be neatly identified as wholly distinct and separate. To a certain extent they influence one another and may be partially overlapping in duration. For example, the existence of certain solution techniques will often influence the construction of the model. Similarly, it is not uncommon for formulation of the problem to undergo periodic refinement even as the subsequent phases of the operations research study are being carried out. Nevertheless, it is reasonably accurate to characterize an operations research study in terms of these five distinct activities. We will discuss each of these phases in turn.

Problem formulation is probably the most difficult matter to discuss in the sense that our remarks must be very general. A problem begins to be formulated when someone or several persons perceive a need that is not currently being met or attempt to state an objective for the future. For example, someone may ask: Where should our company build its new manufacturing plant? Inputs may be forthcoming from various segments of the company. Typical responses would be:

1. "Close to X since they are a big customer" (Marketing).
2. "In the Y area—they are not unionized" (Personnel).
3. "I like A. Power is cheaper there" (Manufacturing).
4. "W is the place. It is cheaper to ship out of there by rail or water" (Marketing and Traffic).

It can be seen that initial guesses and impressions may be in conflict because they represent only partial or fractional insights into the problem. What is the problem? The company must formulate an *objective* when it asks, "Where should we build a plant?" Are we to minimize total cost? Maximize total profit from the plant for one year? For the next ten years? Should we consider locating the plant so as to maximize total company profit? We can see that formulating an objective is most important. This will require extensive discussions between the operations research team and people who have the responsibility to set policy.

Once an objective is formulated, all the variables or factors or considerations, such as those listed in the preceding paragraph, must be carefully identified and any constraints, however vague their statement may be initially, must be taken into consideration. This is a time consuming and often frustrating task but is probably the most important phase of the study. It should be clear that solving the wrong problem with great care is not what we wish to do. The way to avoid solving the wrong problem is to painstakingly search out all the dimensions and variables of the problem, and to formulate as clearly as possible the overall objective of the study.

In Section 1.3 we have discussed some aspects of *model construction*. The general form of an operations research model is given by equations (12). It will be noted that a distinction has been made between controllable variables and those variables that are not controlled. A model that contains controllable variables is sometimes called an *explanatory* model by analogy with laws of science that assume a causative relationship between variables. Models that contain no controllable variables are sometimes called *descriptive* models, in that the model describes the behavior of a system but there is no implication that the uncontrolled variables are causative agents. Statistical correlations are sometimes of this nature. For example, suppose one had collected the kind of data shown in Table 1-1.

TABLE 1-1 Sale of Cars and Planes in City *A*

Year	Number of Cars Sold	Number of Planes Sold
1960	105,008	450
1961	120,121	510
1962	150,080	650
1963	139,850	600
1964	155,021	660
1965	160,205	684
1966	175,103	756
1967	180,020	778

If $x=$ number of planes sold and $y=$ number of cars sold, a very good fit to this data is given by a straight line, i.e., an equation of the form $y = a_0 + a_1x$, where the coefficients a_0 and a_1 can be determined by a least squares regression analysis of the data of Table 1–1. However, it would be a mistake to infer that the sale of planes (the "independent" variable) is *causing* the sale of cars (the "dependent" variable). For one thing, it would be just as easy to analyze the data to yield $x = b_0 + b_1y$; now the roles of the "dependent" and "independent" variables have been reversed. More importantly, the seemingly lawlike regularity exhibited by these equations is merely a reflection of the fact that either the regularity is accidental or one or more *other* variables are causing the behavior exhibited by both x and y and, therefore, the close observed correlation. Nevertheless, descriptive models are often used during data gathering and data analysis to gain insight into which variables should be included in a model.

As an example of how a descriptive model is used, consider the case of a marketing analyst for a computer company who is trying to estimate what the sales potential will be for a new computer the company has developed. It is believed that the computer has features which make it marketable to a certain class of customers. The market analyst then identifies a large number of variables that he believes may have influenced these potential or actual customers in the past when selecting a similar computer. Some of these might include central processor fetch time, word length, ease of remote terminal operation, availability and cost of various software items, and so forth. He may then have a survey conducted among potential customers to see how important these items are considered to be in the cost range under consideration. With this data he can develop a regression model of the form:

$$A = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$$

where A is the relative "attractiveness" of the computer using some composite weighted measure of the accumulated survey data. Each of the $f_j(x_j)$ is some representation that is hoped to be a suitable measure of the contribution of that variable to A. For example, x_1 might be word length and $f_1(x_1)$ might be represented as:

$$f_1(x_1) = a_1 + b_1x_1 + c_1x_1^2$$

In regression analysis, we determine all the coefficients such as a_1, b_1, c_1 for each $f_j(x_j)$ and then, using standard statistical procedures, attempt to determine how much of the variation observed in A is attributable to variations in each of the variables x_j. Such analyses can often yield important information about which variables are significant. If poor correlation is found, it may also indicate that we have not considered all the relevant variables.

There is a strong interaction between the gathering of data and the construction of a model. Often the availability of data or the difficulty in collecting or generating sufficient data for certain kinds of models may affect the complexity and structure of the model. It should be realized that a usable approximate model with modest data requirements is often to be preferred to a more exact model in which the data requirements cannot be met.

The third phase of an operations research study is the *solution of the model.*

This can be accomplished in a number of ways, depending upon the nature of the model and its mathematical complexity.

If a model has a relatively small number of variables and constraints, then so-called classical techniques of mathematical analysis (solution methods that stem from calculus and advanced analysis) can be used to obtain a solution in closed form to the equations and inequalities that comprise it. The number of instances in which this can be done is regrettably small. The simple gear manufacturer's problem in Section 1.2 is an example of such a simple method.

More often an *iterative* solution technique is required to solve model formulations of the type represented by equations (12). An iterative technique is one in which a tentative solution is obtained by some means. It is then examined to see if it is optimal; if not, it provides some information as to how a new candidate solution can be obtained that will be at least as good and probably better. This process is continued until an optimal solution is obtained and recognized. Many of the commonly used algorithms* employed in operations research are iterative in nature. Most of the well-known algorithms of linear, nonlinear and dynamic programming are iterative methods. More will be said of these in later chapters.

To illustrate the nature of an iterative method, we will solve the gear manufacturer problem by simple trial and error, which is an example of an iterative procedure. The expression to be minimized is:

$$S_c = \frac{C_I T y}{2} + \frac{C_s D}{y} \tag{13}$$

Substituting the values of the known constants into equation (13) yields:

$$S_c = \frac{(0.15)(12)y}{2} + \frac{(500)(50{,}000)}{y}$$

which simplifies to:

$$S_c = 0.9y + \frac{25{,}000{,}000}{y}$$

We might guess an initial value of $y = 4000$ and see what this yields for S_c. We then obtain $S_c = \$9850$. We next try other values slightly above and below 4000. For $y = 3900$, $S_c = \$9920.3$ and for $y = 4100$, $S_c = \$9787.6$. This indicates that we need to increase y to make S_c smaller, which is what we wish to do. We next try $y = 4500$ and find that $S_c = \$9605.6$. This is an improvement so we continue to increase y. For $y = 5000$, $S_c = \$9500$. S_c is still decreasing. We next try $y = 5500$ and find that $S_c = \$9495.4$. This is lower than \$9500 so we increase y to 5600 and find that $S_c = \$9504.3$. We see that we have passed the minimum. In fact, we cannot be sure that it is even greater than $y = 5500$. All we know is that it is between $y = 5000$ and $y = 5500$. Further trials in this range would yield the answer we previously found, viz., $y = 5270$ and $S_c = \$9487$.

*An algorithm is a complete step by step description of how to carry out a computational procedure to obtain a solution to some problem.

Another commonly used solution technique is that of *simulation*. By a simulation we mean the evaluation of one or more equations, or a set of functional relationships, in which one or more of the variables are *stochastic* variables. A stochastic variable is one for which the value at any given time is the result of a random selection from some given probability distribution of values for that variable. (See Chapter 10.) For example, suppose we wished to evaluate the average value of $y = \frac{x^2}{1-x}$ where x can be any value between 0 and 0.9 with equal probability, i.e., with probability equal to $\frac{1}{10}$. The usual way to evaluate this would be to calculate:

$$0.1\,\frac{0^2}{1-0} + 0.1\,\frac{(.1)^2}{1-.1} + \ldots + 0.1\,\frac{(.9)^2}{1-.9} = 1.479$$

Let us now consider an alternative method of finding the average value of y using a simulation approach. The method consists of selecting random numbers (from a table or by generation) in the range of 0 to 0.9, with equal probability, and computing an average over whatever sample size is selected. We see the result of this procedure in Table 1–2, where the values of x were chosen from a table of random numbers. After 45 observations we have obtained 1.515 as compared with the exact value of 1.479. With a larger sample size we could approach the true value with an arbitrary degree of accuracy.

This simple example of simulation can be extended to analyses of complex systems for which either equations cannot be solved by any other method or the relationships describing various phases of the operation can be described only in implicit probabilistic terms. Examples of this will be seen in subsequent chapters.

The fourth phase of an operations research study is *testing of the model* after it is solved and *evaluating the solution* obtained. It is obvious that this phase is of great importance and why this is so should be equally obvious. The best trained analysts make errors of commission and omission. This can result in any one or more of the following kinds of deficiencies being present in the system model:

1. The mathematical structure of the model may be in error.
2. Parameters (constants) in the model may be in error.
3. Important variables may have been omitted.
4. Variables may have been included which are irrelevant.

Generally, in order to test the model one can use either *prospective* or *retrospective* tests. If possible, a prospective test, i.e., one in which the actual physical system is operated according to the solution obtained by solving the model, provides a reasonably good test of the structure of the model and some basic parameters. However, this is not always possible. When it is not, we must resort to comparing past performance of the system with the model's predictions and conclusions. This is generally not a simple thing to do. First, we must be sure about what actually *did* happen in the past. This entails knowing the appropriate values of all model parameters during the period of time under consideration, which may not always be readily available. In such a situation it is also important that the past performance be a *representative* sample of the operating system behavior, i.e., it must include variations of operating performance over the fullest range possible.

In testing the appropriateness of including or excluding variables, the sensitivity of the model to small perturbations in the values of measured variables, and so on, it

TABLE 1–2 Estimation by Simulation of $y = \frac{x^2}{1-x}$

x	y	Cumulative Average
0.5	0.5	
0.9	8.1	
0.6	0.9	
0.9	8.1	
0.3	0.129	3.546
0.8	3.2	
0.1	0.011	
0	0	
0.3	0.129	
0.6	0.9	2.197
0.6	0.9	
0	0	
0	0	
0.7	1.633	
0.5	0.5	1.667
0.4	0.267	
0.2	0.050	
0.4	0.267	
0.2	0.050	
0.3	0.129	1.288
0.2	0.050	
0.1	0.011	
0.6	0.9	
0.1	0.011	
0.3	0.129	1.075
0.2	0.050	
0.5	0.5	
0.8	3.2	
0.2	0.05	
0.6	0.9	1.052
0.3	0.129	
0.4	0.267	
0.4	0.267	
0.7	1.633	1.199
0.9	8.1	
0.6	0.9	
0.3	0.129	
0.6	0.9	
0.5	0.5	
0.9	8.1	1.312
0.8	3.2	
0.4	0.267	
0.9	8.1	
0.8	3.2	
0.6	0.9	1.515

is important to use proven standard statistical techniques. The most commonly used methods for this purpose are regression analysis and the analysis of variance and covariance. The operations research study team should have a thorough understanding of sampling methods, experimental design and various methods of correlation and regression analysis.

The last phase of an operations research study is concerned with the *implementation of the solution.* The greatest mistake that can be made on the part of all concerned is to assume that the implementation will automatically occur if the results of the operations research study are turned over to the appropriate managers. Unless the managers themselves have been kept fully informed of the progress of the study and have been directly involved in reviewing and commenting on the progress, their reaction may well be to simply ignore the recommendations of the study. However, if they have participated, themselves or via representatives, in various phases of the study, they will be more inclined to consider implementing the recommendations of the operations research study.

At this point it is very important to be certain that the solution being implemented is for the conditions that *currently* prevail in the system under study. It is not uncommon for many things to change over time. For example, costs for labor or materials may have changed; a competing product may have changed profit margins; the demand for the system output may have changed. It is most important to check *all* the original data and assumptions at the time implementation of the recommended solution is attempted.

If the solution is implemented it is vitally important that what is done is in fact what was recommended. Both inadvertent and deliberate errors can and do occur and one must monitor the implementation phase very carefully to avoid their occurrence.

Lastly, it is important that implementation of the recommended new or changed operation of the system continue for a sufficiently long period of time so that the results are truly representative of the new conditions. Often this can be interpreted only in a statistical sense; hence a sufficiently large sample, in terms of time, may be required.

1.5 BASIC MODEL TYPES OF OPERATIONS RESEARCH

In a sense, no two operations research studies are identical. However, the results of the formulation of a mathematical model for many different kinds of systems often lead to very similar kinds of mathematical representations. A number of general areas have thus been identified and fairly elaborate mathematical and computational methods have come to be associated with these areas. In turn, the existence of these *model types* has led practitioners to consider using them as a guiding framework for the analysis and formulation of a particular problem.

It should be emphasized that the model types we discuss are in many instances only points of departure. They often will not fit exactly a real-world situation to be modeled. However, they do offer us examples of how to proceed in formulating a problem. Then we may proceed to see exactly where the real world differs from the model. This, in turn, may lead to a somewhat more complex but more accurate model formulation.

Some of these model types that have been identified, and which will be discussed in subsequent chapters of this book are:

1. Allocation Models (Chapter 2)
2. Additional Allocation Models (Chapter 4)
3. Network Models (Chapter 5)
4. Location Models (Chapter 6)
5. Scheduling Models (Chapter 7)
6. Probabilistic Decision Models (Chapter 10)
7. Markov Models (Chapter 11)
8. Queueing Models (Chapter 12)
9. Inventory Models (Chapter 13)

Allocation models are generally concerned with how to allocate scarce or constrained resources so as to meet all requirements, both of supply and demand, in such a way as to maximize profit or minimize cost. For example, one might wish to allocate raw materials to a number of different chemical processes so as to satisfy the constraints of how much of each raw material is required for each process and to meet certain minimum sales requirements for each product. The problem is to do this allocation so that the total profit from the sale of each product is maximized.

Network models are a special category of allocation problems in which supply and demand points are connected by a "network" of some kind. The optimal routing of messages over a communication network and the minimum cost pattern of making shipments of material from several sources of supply to many shipping points are both examples of problems that can be solved by using network theory.

Location models are generally concerned with finding the location in space of one or more service facilities (sources) in order to minimize the cost (or maximize the profit) of supplying some service to a set of supply points (sinks). Examples of possible source-sink pairs are:

Source	*Sink*
Warehouses	Retail customers
Fire stations	Areas of a city
Telephone switching center	Telephone customers
Oil refinery	Oil producing wells

Scheduling models are concerned with situations in which not only providing service in a given period of time is important, but the *order* in which the various component services are provided is important. For example, in an assembly plant it is important not only that sufficient time be available on a sanding machine and a spray paint operation but also that they be available in that order, so that total job completion times are met.

Probabilistic decision models are generally concerned with trying to make an optimal decision in the face of imperfect or uncertain knowledge of some of the parameters or variables of the situation to be modeled. For example, how long one should keep a car depends upon variables relating to the amount of maintenance required. This is not known except probabilistically. Hence, any model which describes this situation must include probabilistic variables.

Queueing models are concerned with the problems of providing a service when the jobs or items requiring service arrive at different times and the possibility of a delay exists because of waiting time. One wishes to minimize waiting time or have a reasonably high probability of being able to provide the service within a given period of time. Examples of such situations are traffic toll booths, supermarket check-out counters, arrival of automobiles at repair depots, and so on.

Inventory models are concerned with the amount of required resources to be held in storage for manufacturing or sales, and when and how much should be ordered or produced so as to minimize total cost of some operation.

PROBLEMS

1. You wish to replace an automobile that you currently own. Your objective in doing so is to achieve the lowest cost transportation, assuming you continue to own and drive an automobile. List all the factors (variables) that you think should be considered. Derive a mathematical expression, which expresses cost in terms of the variables, that takes all of these factors into account, even if some of the data are difficult to obtain. This expression is one that, if you were solving the model (do not do so), you would wish to minimize.

2. Consider your university library. What do you think the basic objective of the library is? Try to formulate and state this as clearly as possible. Having done so, try to relate this objective to all the factors or variables relating to the operation of the library. Can this be quantified? In terms of the objective, are there radically different ways the services inherent in the library's objective could be performed? How could you decide how efficiently a library is operating? Relate this question to your previous formulations.

3. There are many areas of human endeavor that are not commonly considered to be "quantifiable," i.e., describable in terms of numbers or mathematical expressions. Suppose we ignore this conventional wisdom. Consider and suggest how one might go about establishing a quantification scheme for how well any individual expresses his thoughts "clearly" and "interestingly" when he puts them in writing.

4. You have a small company that produces and sells doughnuts to groceries. Describe each element of cost that needs to be considered in calculating your expenses and profits. Suppose a competitor enters the market and sells the doughnuts for two cents per dozen less than yours. In order to hold your profit constant, consider what elements of cost reduction are available to you. Describe as much of this in numerical terms as possible.

5. Suppose you are given the problem of how to increase the efficiency of a single post office. It is suggested that this can be accomplished by increased automation of all or many of the functions performed by people. Describe in detail the methodology you would employ to try to decide whether or not increased use of automation would accomplish the desired objective.

6. As in most things, there are two "schools of thought" concerning where one should locate a new grocery store. One group holds that it should not be too close to a competitor's store. The other holds that it should be as close as possible. What would you do to decide this issue? Explain what experiments and analysis you would undertake. What other factors, besides closeness to a competitor, do you think should influence this decision? How might you consider *all* of these factors simultaneously?