Detecting Infeasible Problems

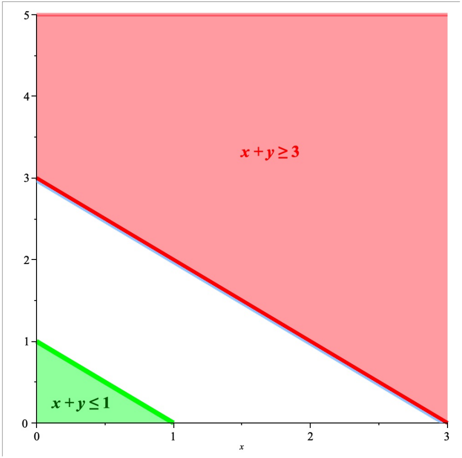
Class 11

March 8, 2023

Announcements

Exam 1: Wednesday, March 15 at 7 PM

How Does Simplex Method Detect InfeasibleProblems?



How the Simplex Method Recognizes INFEASIBLE Problems

Maximize $Z = 2x + 3y$

subject to

1) $1x + 1y \geq 3$

2) $1x + 1y \leq 1$

and $x \geq 0$, $y \geq 0$.

Step 1: Subtract surplus variables from first constraint and add a slack variable to the second constrained

Maximize $Z = 2x + 3y$

subject to

1) $1x + 1y - u = 3$

2) $1x + 1y + v = 1$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$

$$\text{Maximize } Z = 2x + 3y$$

subject to

$$\begin{aligned} 1)\quad & 1x + 1y - u = 3 \\ 2)\quad & 1x + 1y + v = 1 \end{aligned}$$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$

Step 2: Add artificial variables to first constraint to generate a basic feasible solution

$$\text{Maximize } Z = 2x + 3y$$

subject to

$$\begin{aligned} 1)\quad & 1x + 1y - u + a = 3 \\ 2)\quad & 1x + 1y + v = 1 \end{aligned}$$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$, $a \geq 0$

Maximize $Z = 2x + 3y$

subject to

1)	$1\,x$	+	$1\,y$	−	u		+ a	=	3
2)	$1\,x$	+	$1\,y$			+v		=	1

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$, $a \geq 0$

Step 3: Add **penalty** for using the artificial variable:

Maximize $Z = 2x + 3y - Ma$

Maximize Z = $2x + 3y - Ma$
subject to

1) $1x + 1y - u + a = 3$
2) $1x + 1y + v = 1$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$, $a \geq 0$

FORM TABLEAU:

	Z	*x*	*y*	*u*	*a*	*v*		
	1	−2	−3	0	*M*	0	=	0
a	0	1	3	−1	1	0	=	3
v	0	38	34	0	0	1	=	50

Make ***a*** column basic by subtracting M * (*a* row) from objective function row.

Solve Interactively by the Simplex Method:

Bas Var	Eq No	Z	Coefficient of x	y	u	a	v	Right side
			-1M	-1M	1M			-3M
Z	**0**	**1**	**- 2**	**- 3**	**+ 0**	**0**	**0**	**0**
a	1	0	1	1	-1	1	0	3
v	2	0	1	[1]	0	0	1	1

Bas Var	Eq No	Z	*x*	*y*	*u*	*a*	*v*	Right side
					1M		**1M**	**-2M**
Z	**0**	**1**	**1**	**0 +**	**0**	**0 +**	**3**	**3**
a	1	0	0	0	-1	1	-1	2
y	2	0	1	1	0	0	1	1

We have reached a basic feasible optimal solution of the problem

BUT

it has one of the artificial variables as a basic variable with a positive value.

A Problem With Mixed Constraints

Maximize $2x - y + 3z$ subject to

$$2y + z \leq 2$$

$$x + y + z = 4$$

$$x - 2y + z \geq 3$$

$$x \geq 0, y \geq 0, z \geq 0$$

Introduce slack variable u to $\leq$ constraint
Introduce artificial variable a to $=$ constraint
Introduce slack variable v and artificial variable b to $\geq$ constraint.

Maximize $2x - y + 3z$ subject to

$$2y + z + u = 2$$

$$x + y + z + a = 4$$

$$x - 2y + z - v + b = 3$$

$$x \geq 0, y \geq 0, z \geq 0$$

Tableau for the Optimal Basic Feasible Solution of Primal

	Z	x	y	z	u	a	v	b	
Z	1	-2	1	-3	0	-M	0	-M	
u	0	0	2	1	1	0	0	0	2
a	0	1	1	1	0	1	0	0	4
b	0	1	-2	1	0	0	-1	1	3

Using IOR to Solve Mixed Constraint Problem (Annotated)

```
  Linear Programming Model:

  Number of Decision Variables:      3
  Number of Functional Constraints:  3
  Max Z =      2 X1 -    1 X2 +    3 X3

  subject to

   1)          0 X1 +    2 X2 +    1 X3 <=       2

   2)          1 X1 +    1 X2 +    1 X3  =       4

   3)          1 X1 -    2 X2 +    1 X3 <=       3

  and

          X1 >= 0, X2 >= 0, X3 >= 0.
```

Introduce X4 as Slack Variable in Constraint 1
Introduce X5 as Artificial Variable in Constraint 2
Introduce X6 as Surplus Variable and x7 as Artificial Variable in Constraint 3

Solve Interactively by the Simplex Method:

Initial Tableau:

```
  Bas|Eq|                     Coefficient of                   | Right
  Var|No| Z|   X1     X2     X3     X4     X5     X6     X7    |  side
  ___|__|__|___________________________________________________|______
     |  |  | -2M     1M    -2M                   1M            |   -7M
   Z | 0| 1|-   2 +    1 -    3      0      0 +    0      0    |     0
   X4| 1| 0|    0      2     [1]      1      0      0     0    |     2
   X5| 2| 0|    1      1      1      0      1      0      0    |     4
   X7| 3| 0|    1     -2      1      0      0     -1      1    |     3
```

Not optimal because negative numbers in Z row. Most negative is -2M-3

X3 will enter basis and X4 will leave
ITERTAION 1

Resulting Tableau:

```
Bas|Eq|                   Coefficient of                   | Right
Var|No| Z|   X1    X2    X3    X4    X5    X6    X7   |  side
___|__|__|_____________________________________________|______
   |  |  |  -2M    5M          2M          1M          |   -3M
 Z | 0| 1|-   2 +   7     0 +   3     0 +   0     0    |     6
 X3| 1| 0|    0     2     1     1     0     0     0    |     2
 X5| 2| 0|    1    -1     0    -1     1     0     0    |     2
 X7| 3| 0|    [1]   -4     0    -1     0    -1     1   |     1
```

Not optimal because negative number in Z row: -2M-2.

X1 will enter basis and X7 will leave.
We will drive one of the artificial basis variables (X7)
out of the basis.
ITERTAION 2

Resulting Tableau:

```
Bas|Eq|                  Coefficient of                  | Right
Var|No| Z|   X1    X2    X3    X4    X5    X6    X7    |  side
___|__|__|_____________________________________________|______
   |  |  |        -3M                      -1M    2M    |   -1M
 Z | 0| 1|    0 -   1     0     1     0 -   2 +   2    |     8
 X3| 1| 0|    0     2     1     1     0     0     0    |     2
 X5| 2| 0|    0     [3]    0     0     1     1    -1   |     1
 X1| 3| 0|    1    -4     0    -1     0    -1     1    |     1
```

Not optimal because negative numbers in Z row, Most negative is -3M+1
X2 will enter the basis and the artificial variable X5 will leave.
ITERTAION 3

Resulting Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
							1M		1M	
Z	0	1	0	0	0	1	+0.33	−1.67	+1.67	8.333
X3	1	0	0	0	1	1	−0.67	−0.67	0.667	1.333
X2	2	0	0	1	0	0	0.333	[0.333]	−0.33	0.333
X1	3	0	1	0	0	−1	1.333	0.333	−0.33	2.333

The artificial variables are gone!
However, not an optimal solution because of the −1.67 in Z row
X6 will enter the basis and X2 will leave.
ITERTAION 4

Resulting Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	5	0	1	1M + 2	0	1M + 0	10
X3	1	0	0	2	1	1	0	0	0	2
X6	2	0	0	3	0	0	1	1	−1	1
X1	3	0	1	−1	0	−1	1	0	0	2

WE HAVE AN OPTIMAL BASIC FEASIBLE SOLUTION.
THERE ARE NO NEGATIVE NUMBERS IN Z ROW.
THE OPTIMAL SOLUTION OF THE ORIGINAL PROBLEM Is
x (X1) = 2
y (X2) = 0
z (X3) = 2
MAXIMUM VALUE OF OBJECTIVE FUNCTION IS 10.

No Non-negativity Constraints?

Exercise 4.6 - 14ab

There are no non-negativity constraints on the decision variables. The Simplex Method, however, is based on keeping all variables ≥ 0.

What To Do?

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What To Do? Observe: Every real number can be written as the

difference of two non-negative numbers.

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Example: -5 = 3 - 8.

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What To Do? Observe: Every real number can be written as the

difference of two non-negative numbers.

Example: -5 = 3 - 8.

Thus write $x = x^+ - x^-$ and $y = y^+ - y^-$ so a constraint of the form $30x + 11y \leq 12$ becomes $30x^+ - 30x^- + 11y^+ - 11y^- \leq 12$, etc.

The Simplex Method So Far

Use: Solve Linear Programming Problems: maximize linear function subject to linear constraints.

Features:

- Can detect if no feasible solution exists.
- Can find an initial basic feasible solution (bfs) if the problem is feasible.
- Can tell if the current bfs solution is optimal.
- Can reveal, when current bfs is optimal, if multiple optimal solutions exist.
- Provides a way, if the current bfs is not optimal, to obtain a new bfs with a better objective function value.
- Proceeds by a sequence of iterations, each of which involves putting one currently nonbasic variable into the basis and removing one currently basic one.
- Uses only simple arithmetic and elementary row operations at each step.

Drawbacks of the Simplex Method

- Cycling can occur if degeneracy happens. (Solution: use Bland's Rule or another simple modification to prevent cycling).
- Although Simplex Method generally runs to completion quickly, it may in the worst possible cases visit every bfs before reaching the optimal one. (An inherent limitation!)

What Questions Remain?

Examine Final Tableaux of Fromage and Cheese Buyer Problems:

	Z	x	y	u	v	w	
Z	1	0	0	0	**5/12**	**1/12**	1250
x	0	1	0	0	**1/6**	**-1/6**	100
y	0	0	1	0	**-1/12**	**5/24**	200
u	0	0	0	1	**-4**	**5/2**	600

What is the meaning of the green numbers?

	Z	x	y	z	u	v	
Z	1	**600**	0	0	**100**	**200**	-1250
y	0	**4**	1	0	**-1/6**	**1/12**	**5/12**
z	0	**-5/2**	0	1	**1/6**	- **-5/24**	**1/12**