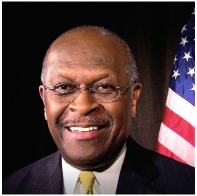
Some Linear Algebra Behind Simplex Method;
Artificial Variables and Initial Basic Feasible Solutions

Class 10

March 6, 2023

Careers for OR Majors



Herman Cain

December 13, 1945– July 30, 2020

BA Mathematics (Morehouse)

MS (Purdue)

American author, business executive (Burger King, Godfather's Pizza, Pillsbury),chair of Federal Reserve Bank of Kansas City, radio host, syndicated columnist, and Tea Party activist from Georgia. He was a candidate for the 2012 U.S. Republican Party presidential nomination.

Notes on Assignment 3
Some Linear Algebra Behind the Simplex Method (online)

Some Linear Algebra Behind the Simplex Method

Original Problem: n decision variables, m constraints

Fromage: $n = 2$, $m = 3$
Chairs: $n = 3$, $m = 3$

Augment with m slack variables so we can represent constraint set as the solution set of a system of linear equations with $(n + m)$ variables and m equations.

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

where we can write **A as**

$$\mathbf{A} = (\mathbf{B}, \mathbf{N})$$

where **B** is an $\boldsymbol{m}$ by $\boldsymbol{m}$ invertible matrix and

$$\vec{\mathbf{x}} = \begin{pmatrix} \overrightarrow{\mathbf{x}_B} \\ \overrightarrow{\mathbf{x}_N} \end{pmatrix}$$

For Original Fromage:

$$\mathbf{A} = \begin{array}{c} \begin{matrix} u & v & w & x & y \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 30 & 12 \\ 0 & 1 & 0 & 10 & 8 \\ 0 & 0 & 1 & 4 & 8 \end{pmatrix} \end{array} \quad \text{and} \quad \vec{\mathbf{x}} = \begin{pmatrix} u \\ v \\ w \\ x \\ y \end{pmatrix}$$

Then we can write

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

as

$$(\mathbf{B},\mathbf{N}) \begin{pmatrix} \overrightarrow{\mathbf{x}_{\mathbf{B}}} \\ \overrightarrow{\mathbf{x}_{\mathbf{N}}} \end{pmatrix} = \vec{\mathbf{b}}$$

$$\mathbf{B}\overrightarrow{\mathbf{x}_{\mathbf{B}}} + \mathbf{N}\overrightarrow{\mathbf{x}_{\mathbf{N}}} = \vec{\mathbf{b}}$$

$$\mathbf{B}\overrightarrow{\mathbf{x}_{\mathbf{B}}} = \vec{\mathbf{b}} - \mathbf{N}\overrightarrow{\mathbf{x}_{\mathbf{N}}}$$

$$\overrightarrow{\mathbf{x}_B} = \mathbf{B}^{-1}\vec{\mathbf{b}} - \mathbf{B}^{-1}\mathbf{N}\overrightarrow{\mathbf{x}_N}$$

A *basic* solution is one in which

$$\overrightarrow{\mathbf{x}_N} = \vec{\mathbf{0}}$$

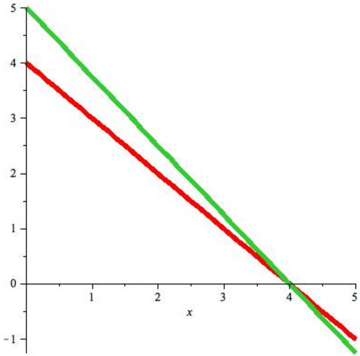
A *basic feasible solution* is a basic solution if

$$\mathbf{B}^{-1}\vec{\mathbf{b}} \geq \vec{\mathbf{0}}$$

The calculations are easy if **B** is the identity matrix.

Example. Suppose the constraint set is given by

$$x + \quad y \leq 4$$
$$5\,x + 4\,y \leq 20$$
$$x \geq 0,\ y \geq 0$$



Convert to equations

$$\begin{array}{rcrcrcrcr} 1x & + & 1y & + & 1u & & & = & 4 \\ 5x & + & 4y & & & + & 1v & = & 20 \end{array}$$
$$x,\ y,\ u,\ v \text{ all} \geq 0$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{pmatrix} \quad \text{(columns } x, y, u, v\text{)}$$

and

$$\vec{\mathbf{b}} = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

Let

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 5 & 4 \end{pmatrix} \quad \text{(columns } x, y\text{)}$$

Then

$$\mathbf{B}^{-1}\vec{\mathbf{b}} = \begin{pmatrix} -4 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 20 \end{pmatrix} = \begin{pmatrix} -16+20 \\ 20-20 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

so the basic feasible solution is $x = 4$, $y = 4$, $u = 0$, $v = 0$.

But we also could have chosen

$$\hat{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \quad \text{(columns } x, v\text{)}$$

where y, u are the nonbasic variables. Here

$$\hat{\mathbf{B}}^{-1}\vec{\mathbf{b}} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 20 \end{pmatrix} = \begin{pmatrix} 4+0 \\ -20+20 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

The extreme points corresponding to $\mathbf{B}$ and $\hat{\mathbf{B}}$ might be identical.

Initial Basic Feasible Solution

Big M Method

Initial Basic Feasible Solution

How Do We Obtain an Initial Basic Feasible Solution?

Case 1: Constraint of the form $\leq$ Positive Number

Example: $3x + 5y + 7z \leq 276$

Introduce **slack variable**: $3x + 5y + 7z + u = 276$
Initial Solution? Set $x = 0, y = 0, z = 0, u = 276$
u is a basic variable

Initial Basic Feasible Solution

How Do We Obtain an Initial Basic Feasible Solution?

Case 2: Constraint of the form $\geq$ Positive Number

Example: $3x + 5y + 7z \geq 276$

Introduce **surplus variable**: $3x + 5y + 7z - u = 276$
Introduce **artificial variable**: $3x + 5y + 7z - u + a = 276$

Initial Solution? Set $x = 0, y = 0, z = 0, u = 0, a = 276$
a is a basic variable

Sherry's Breakfast Problem

Minimize $4x + 6.5y$

subject to

$$\begin{aligned} 1)\quad 1x + 3y &\geq 3 \qquad \text{(iron)} \\ 2)\quad 38x + 34y &\geq 50 \qquad \text{(protein)} \end{aligned}$$

and $x \geq 0, y \geq 0.$

Step 1. Convert to Maximization Problem

$$\text{Maximize } Z = -4x - 6.5y$$

Step 2: Subtract *surplus* variables from each constraint:

$$\text{Maximize } Z = -4x - 6.5y$$

subject to

$$\begin{aligned} 1)\quad 1x + 3y - u &= 3 \\ 2)\quad 38x + 34y - v &= 50 \end{aligned}$$

and $x \geq 0, y \geq 0, u \geq 0, v \geq 0$

Step 3: Add artificial variables to each constraint to generate a basic feasible solution

$$\text{Maximize } Z = -4x - 6.5y$$

subject to

$$\begin{aligned} &1)\ \ 1x + 3y - u + a = 3 \\ &2)\ \ 38x + 34y - v + b = 50 \end{aligned}$$

and $x \geq 0, y \geq 0, u \geq 0, v \geq 0, a \geq 0, b \geq 0$

Step 4. Adjust the objective function to make use of artificial variables prohibitively expensive

$$\text{Maximize } Z = -4x - 6.5y - pa - pb$$

OR

$$\text{Maximize } Z = -4x - 6.5y - Ma - Mb$$

Where p (or M) is an unspecified by very large positive number, the penalty for using one of these artificial variables.

Cheese Buyer's Problem

Cheese Buyer's Problem

Fromage Cheese Company Problem

Problem: Maximize $Z = 4.5x + 4y$

subject to constraints

$$30x + 12y \leq 6000 \text{ (Cheddar)}$$
$$10x + 8y \leq 2600 \text{ (Swiss)}$$
$$4x + 8y \leq 2000 \text{ (Brie)}$$
$$x, y \geq 0$$

Cheese Buyer's Problem

Minimize $Z = 6000x + 2600y + 2000z$

subject to constraints

$$30x + 10y + 4z \geq 4.5$$
$$12x + 8y + 8z \geq 4$$
$$x, y, z \geq 0$$

Convert To Equations

Cheese Buyer's Problem

$$\text{Maximize } Z = -6000x - 2600y - 2000z - Ma - Mb$$

subject to constraints

$$30x + 10y + 4z - u + a = 9/2$$

$$12x + 8y + 8z - v + b = 4$$

$$x, y, z, u, v, a, b \geq 0$$

Initial tableau

	Z	x	y	z	u	v	a	b	
Z	1	6000	2600	2000	0	0	M	M	0
a	0	30	10	4	-1	0	1	0	$9/2$
b	0	12	8	8	0	-1	0	1	4

Need to make a and b columns basic.

Subtract M times Second Row from Objective Function Row

Subtract M times Third Row from Objective Function Row

After Making *a* and *b* Columns Basic

	Z	x	y	z	u	v	a	b	
Z	1	6000-42M	2600-18M	2000 -12M	M	M	0	0	$-\frac{17M}{2}$
a	0	30	10	4	-1	0	1	0	$9/2$
b	0	12	8	8	0	-1	0	1	4

x will enter the basis.

θ ratios are $(9/2)/(30) = 3/20$ and $4/12 = 1/3$.

Pivot Entry is 30; a will leave the basis.

Row Operations:

Divide Row 2 by 30

Add (-6000+42) Times Row 2 to Objective Function Row.

Add (-12) Times Row 2 to Row 3.

After Making a and b Columns Basic

	Z	x	y	z	u	v	a	b	
Z	1	6000-42M	2600-18M	2000 -12M	M	M	0	0	$-\frac{17M}{2}$
a	0	30	10	4	-1	0	1	0	$9/2$
b	0	12	8	8	0	-1	0	1	4

x will enter the basis.

θ ratios are $(9/2)/(30) = 3/20$ and $4/12 = 1/3$.

Pivot Entry is 30; a will leave the basis.

Row Operations:

Divide Row 2 by 30

Add (-6000+42) Times Row 2 to Objective Function Row.

Add (-12) Times Row 2 to Row 3.

Tableau After First Iteration

	Z	x	y	z	u	v	a	b	
Z	1	0	600-4M	1200-$\frac{32}{5}$M	200-$\frac{2}{5}$M	M	-200+$\frac{7}{5}$M	0	-900 -$\frac{11}{5}$M
x	0	1	1/3	2/15	-1/30	0	1/30	0	3/20
b	0	0	4	32/5	2/5	-1	-2/5	1	11/5

z will enter the basis.

θ ratios are $(3/20)/(2/15) = 9/8$ and $(11/5)/(32/5) = 11/32$.

Pivot Entry is 32/5; b will leave the basis.

Row Operations:

Divide Row 3 by 32/5

Add (-1200 + 32M/5) Times Row 3 to Objective Function Row.

Add (-2/15) Times Row 3 to Row 2.

Tableau After Second Iteration

	Z	x	y	z	u	v	a	b	
Z	1	0	-150	0	125	375/2	M - 125	M - 375/2	$-\frac{2625}{2}$
x	0	1	1/4	0	-1/24	1/48	1/24	-1/48	5/48
z	0	0	5/8	1	1/16	-5/32	-1/16	5/32	11/32

y will enter the basis.

θ ratios are $(5/48)/(1/4) = 5/12$ and $(11/32)/(5/8) = 11/20$.

Pivot Entry is 1/4; x will leave the basis.

Row Operations:

Multiply Row 2 by 4

Add 150 Times Row 2 to Objective Function Row.

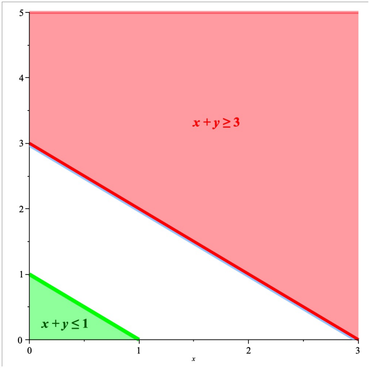
Add (-5/8) Times Row 2 to Row 3.

Tableau After Third Iteration

	Z	x	y	z	u	v	a	b	
Z	1	600	0	0	100	200	M - 100	M - 200	-1250
y	0	4	1	0	-1/6	1/12	1/6	-1/12	5/12
z	0	-5/2	0	1	1/6	-5/24	-1/6	5/24	1/12

We have an optimal basic feasible solution to the problem.
Offer: 5/12 of a dollar for each ounce of Swiss
1/12 of a dollar for each ounce of Brie
0 for each ounce of Cheddar

How Does Simplex Method Detect Infeasbile Problems?



How the Simplex Method Recognizes INFEASIBLE Problems

Maximize $Z = 2x + 3y$

subject to

1) $1x + 1y \geq 3$

2) $1x + 1y \leq 1$

and $x \geq 0$, $y \geq 0$.

Step 1: Subtract surplus variables from first constraint and add a slack variable to the second constrained

Maximize $Z = 2x + 3y$

subject to

1) $1x + 1y - u = 3$

2) $1x + 1y + v = 1$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$

Maximize $Z = 2x + 3y$

subject to

1) $1x + 1y - u = 3$
2) $1x + 1y + v = 1$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$

Step 2: Add artificial variables to first constraint to generate a basic feasible solution

Maximize $Z = 2x + 3y$

subject to

1) $1x + 1y - u + a = 3$
2) $1x + 1y + v = 1$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$, $a \geq 0$

Maximize $Z = 2x + 3y$

subject to

$$
\begin{aligned}
&1)\quad 1x + 1y - u + a = 3\\
&2)\quad 1x + 1y + v = 1
\end{aligned}
$$

and $x \geq 0,\ y \geq 0,\ u \geq 0,\ v \geq 0,\ a \geq 0$

Step 3: Add **penalty** for using the artificial variable:

Maximize $Z = 2x + 3y - Ma$

Maximize Z = $2x + 3y - Ma$
subject to

1) $1x + 1y - u + a = 3$
2) $1x + 1y + v = 1$

and $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$, $a \geq 0$

FORM TABLEAU:

	Z	*x*	*y*	*u*	*a*	*v*		
	1	-2	-3	0	*M*	0	=	0
a	0	1	3	-1	1	0	=	3
v	0	38	34	0	0	1	=	50

Make ***a*** column basic by subtracting M * (*a* row) from objective function row.

Solve Interactively by the Simplex Method:

Bas Var	Eq No	Z	x	y	u	a	v	Right side
			-1M	-1M	1M			-3M
Z	**0**	**1**	**- 2**	**- 3**	**+ 0**	**0**	**0**	**0**
a	1	0	1	1	-1	1	0	3
v	2	0	1	[1]	0	0	1	1

Bas Var	Eq No	Z	x	y	u	a	v	Right side
					1M		**1M**	**-2M**
Z	**0**	**1**	**1**	**0 +**	**0**	**0 +**	**3**	**3**
a	1	0	0	0	-1	1	-1	2
y	2	0	1	1	0	0	1	1

We have reached a basic feasible optimal solution of the problem

BUT

it has one of the artificial variables as a basic variable with a positive value.

The Simplex Method So Far

Use: Solve Linear Programming Problems: maximize linear function subject to linear constraints.

Features:

- Can detect if no feasible solution exists.
- Can find an initial basic feasible solution (bfs) if the problem is feasible.
- Can tell if the current bfs solution is optimal.
- Can reveal, when current bfs is optimal, if multiple optimal solutions exist.
- Provides a way, if the current bfs is not optimal, to obtain a new bfs with a better objective function value.
- Proceeds by a sequence of iterations, each of which involves putting one currently nonbasic variable into the basis and removing one currently basic one.
- Uses only simple arithmetic and elementary row operations at each step.

Drawbacks of the Simplex Method

- Cycling can occur if degeneracy happens. (Solution: use Bland's Rule or another simple modification to prevent cycling).
- Although Simplex Method generally runs to completion quickly, it may in the worst possible cases visit every bfs before reaching the optimal one. (An inherent limitation!)

What Questions Remain?

Examine Final Tableaux of Fromage and Cheese Buyer Problems:

	Z	x	y	u	v	w	
Z	1	0	0	0	**5/12**	**1/12**	1250
x	0	1	0	0	**1/6**	**-1/6**	100
y	0	0	1	0	**-1/12**	**5/24**	200
u	0	0	0	1	**-4**	**5/2**	600

What is the meaning of the green numbers?

	Z	x	y	z	u	v	
Z	1	**600**	0	0	**100**	**200**	-1250
y	0	**4**	1	0	**-1/6**	**1/12**	**5/12**
z	0	**-5/2**	0	1	**1/6**	- **-5/24**	**1/12**