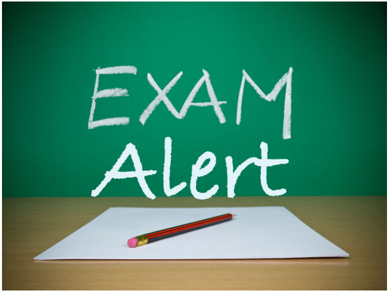
The Simplex Method Continued

Class 8

March 1, 2023

Announcements

Exam 1: Wednesday March 15



Comments on Homework Assignment 2

- Define Variables
  Example: Let L be the number of Lounge Chairs produced in a week
- Write Constraints in the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n \leq b$$

  Left hand side should be a linear combination of the decision variables
  You may have $=$ or $\geq$ in place of $\leq$.
- Do Not Forget Non-Negativity Constraints
- Read problems carefully. Provide justifications for claims. Give examples where asked. Use methods stated.

Review: Behavior of Convex Sets

The definitions and theorems are applicable to sets in R^n, no matter how big n is.

In particular:

The Intersection of Any Collection of Convex Sets is Convex

Solutions of Any Linear Inequality is Convex

Constraint Set of any LP Problem is Convex

A Local Maximum for a Linear Function on a Convex Set is a Global Maximum

The Simplex Method In a Nutshell

Convert Inequalities to Equations Using Slack Variables.
Create Tableau of Coefficients of the Equations.

Step 1

Start with a feasible tableau, i.e. a basic solution that is also non-negative.

Test For Optimality:

If the objective function row of the tableau has no negative entries, the solution is optimal.

Step 2

If there is at least one negative entry, choose as the pivot column the one with the largest negative entry in the objective function row, ignoring the last column.

Step 3

For each row except the objective function row, compute θ ratios for all positive entries in the pivot column

$$\theta = \frac{\text{entry in last column}}{\text{entry in pivot column}}$$

θ ratios must all be non-negative.

Step 4

Choose as the pivot row the one with the smallest θ ratio. If there is a tie, decide arbitrarily.

Step 5

Carry out a pivot operation.

Make Pivot Element $= 1$.

Make Other Elements in Pivot Column $= 0$.

Step 6

Repeat steps 2 through 5 until no new pivot column can be found.
A calculation with the simplex method terminates when either no pivot column or pivot row can be found.
If there is no pivot column (no negative entry in the objective function row), then the current solution is optimal.
If there is a pivot column but all its entries are either zero or negative, then one can show that the problem has an unbounded solution, i.e., the objective function can be made arbitrarily large.

George B. and Anne S. Dantzig
August 1936 Geroge: November 8, 1914 – May 13, 2005
Anne: February 20, 1917 – August 10, 2006

GOAL: FIND A SOLUTION ALGORITHM

which will

1. Find an optimal feasible solution, if it exists, in an efficient manner.
2. Tell us if the objective function is unbounded.
3. Tell us if the constraint set is empty.

The Simplex Method So Far

- Given an initial basic feasible solution, it finds an optimal solution if one exists. (**Does it always?**)
- Each step involves elementary calculations
- It recognizes unbounded problems

How Simplex Method Recognizes Unbounded Problem

Suppose tableau looks like

	Z	x	y	u	v	w	
Z	1	0	0	$-1/30$	$11/20$	0	1230
x	0	1	0	-0	$-1/10$	0	140
y	0	0	1	$-1/12$	$1/4$	0	150
w	0	0	0	$-2/5$	$-8/3$	1	240

If there is a pivot column but all its entries are either zero or negative, then the problem has an unbounded solution, i.e., the objective function can be made arbitrarily large

How Simplex Method Recognizes Unbounded Problem

Suppose tableau looks like

	Z	x	y	u	v	w	
Z	1	0	0	$-1/30$	$11/20$	0	1230
x	0	1	0	-0	$-1/10$	0	140
y	0	0	1	$-1/12$	$1/4$	0	150
w	0	0	0	$-2/5$	$-8/3$	1	240

Corresponding equations are

$$\begin{array}{c} Z = 1230 + \frac{1}{30}u - \frac{11}{20}v \\ \hline x = 140 + \frac{1}{10}v \\ y = 150 + \frac{1}{12}u - \frac{1}{4}v \\ w = 240 + \frac{2}{5}u + \frac{8}{3}w \end{array}$$

If there is a pivot column but all its entries are either zero or negative, then the problem has an unbounded solution, i.e., the objective function can be made arbitrarily large

Other Features of the Simplex Method To Explore

- Does it recognize multiple optimal solutions when they occur?
- Can it detect infeasible problems?
- What might go wrong?

How Does Simplex Method Detect Multiple Optimal Solutions?

Simplex Method Recognizes Multiple Optimal Solutions

$$\text{Maximize } Z = 5x + 4y \text{ subject to}$$
$$30x + 12y \leq 6000$$
$$10x + 8y \leq 2600$$
$$4x + 8y \leq 2000$$
$$x \geq 0, y \geq 0$$

Introduce Slack Variables u, v, w**:**

$$\text{Maximize } Z = 5x + 4y \text{ subject to}$$
$$30x + 12y + u = 6000$$
$$10x + 8y + v = 2600$$
$$4x + 8y + w = 2000$$
$$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0$$

Iteration 1: x **enters basis,** u **leaves.**

Iteration 2: y **enters basis,** v **leaves.**

Simplex Method Recognizes Multiple Optimal Solutions

After 2 iterations, tableau looks like

	Z	x	y	u	v	w	
Z	1	0	0	0	$1/2$	0	1300
x	0	1	0	$1/15$	$-1/10$	0	140
y	0	0	1	$-2/25$	$1/4$	0	150
w	0	0	0	$2/5$	$-8/5$	1	240

We have an optimal basic feasible solution with $x = 140, y = 150$ and $Z = 1300$.

If we try to let v enter the basis, Z will decrease.

But what if u enters the basis?

Z will remain at 1300.

Simplex Method Recognizes Multiple Optimal Solutions

	Z	x	y	u	v	w		
Z	1	0	0	0	$1/2$	0	1300	
x	0	1	0	$1/15$	$-1/10$	0	140	$\frac{140}{1/15} = 2089.6$
y	0	0	1	$-2/25$	$1/4$	0	150	
w	0	0	0	$[2/5]$	$-8/5$	1	240	$\frac{240}{2/5} = 600$

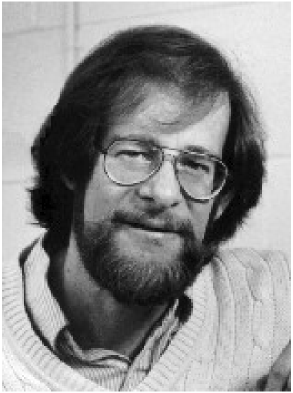
If u enters, then w leaves

Resulting Tableau is

	Z	x	y	u	v	w	
Z	1	0	0	0	$1/2$	0	1300
x	0	1	0	0	$1/6$	-0.17	100
y	0	0	1	0	-0.8	.208	200
u	0	0	0	1	-4	2.5	600

We have another optimal basic feasible solution with $x = 100, y = 200$ and $Z = 1300$.

Degeneracy and Cycling



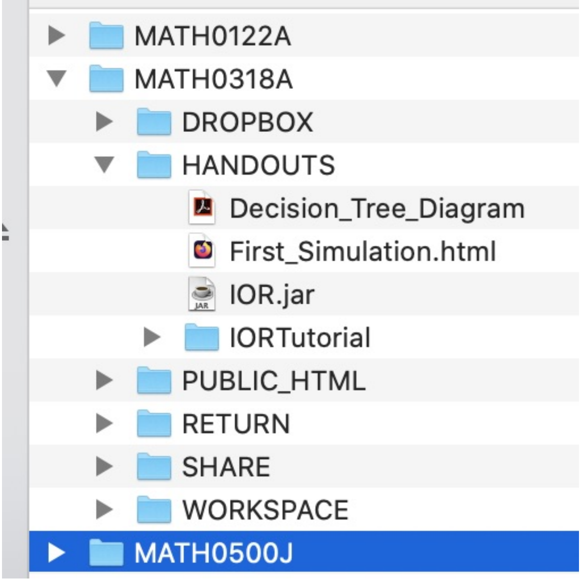
Robert Gary Bland (born February 25, 1948)

Degeneracy

Cycling

Let's see how to use IOR Tutorial

Finding IOR Tutorial



The Simplex Method

Fromage Cheese Company Problem

$$\begin{aligned} -4.5x - 4y + Z &= 0 \\ 30x + 12y + u &= 6000 \\ 10x + 8y + v &= 2600 \\ 4x + 8y + w &= 2000 \end{aligned}$$

Initial Feasible Solution:
$x = 0, y = 0, u = 6000, v = 2500, w = 2000$

	Z	x	y	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

	Z	x	y	u	v	w		
Z	1	-4.5	-4	0	0	0	0	
u	0	30	12	1	0	0	6000	$6000/30 = 200$
v	0	10	8	0	1	0	2600	$2600/10 = 260$
w	0	4	8	0	0	1	2000	$2000/4 = 500$

	Z	x	y	u	v	w		
Z	1	-4.5	-4	0	0	0	0	
u	0	**30**	12	1	0	0	6000	$6000/30 = \mathbf{200}$
v	0	10	8	0	1	0	2600	$2600/10 = 260$
w	0	4	8	0	0	1	2000	$2000/4 = 500$
		$\uparrow$						

Before Iteration:

	Z	x	y	u	v	w		
Z	1	-4.5	-4	0	0	0	0	
u	0	**30**	12	1	0	0	6000	$6000/30 = \mathbf{200}$
v	0	10	8	0	1	0	2600	$2600/10 = 260$
w	0	4	8	0	0	1	2000	$2000/4 = 500$
		$\uparrow$						

After iteration:

	Z	x	y	u	v	w	
Z	1	0	$-11/5$	$3/20$	0	0	900
x	0	1	$2/5$	$1/30$	0	0	200
v	0	0	4	$-1/3$	1	0	600
w	0	0	$32/5$	$-2/15$	0	1	1200

The tableau after several iterations:

	Z	x	y	u	v	w	
Z	1	0	0	$-1/30$	$11/20$	0	1230
x	0	1	0	$1/15$	$-1/10$	0	140
y	0	0	1	$-1/12$	$1/4$	0	150
w	0	0	0	$2/5$	$-8/3$	1	240

Current values: $x = 140, y = 150, w = 240, u = 0, v = 0$

Basic variables: x, y, w

Nonbasic variables: u, v

u will enter the basis.

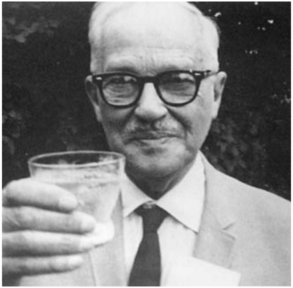
$$
\begin{aligned}
Z\text{-}\mathbf{1/30\ u} + 11/20v &= 1230 \\
x + \mathbf{1/15\ u} - 1/10v &= 140 \\
y\text{-}\mathbf{1/12\ u} + 1/4v &= 150 \\
\mathbf{2/5\ u} - 8/3v + w &= 240
\end{aligned}
$$

Current values: $x = 140, y = 150, w = 240, u = 0, v = 0$
Basic variables: x, y, w
Nonbasic variables: u, v
u will enter the basis.

$$
\begin{aligned}
Z\text{-}\mathbf{1/30\ u} + 11/20v &= 1230 \\
x + \mathbf{1/15\ u} - 1/10v &= 140 \\
y\text{-}\mathbf{1/12\ u} + 1/4v &= 150 \\
\mathbf{2/5\ u} - 8/3v + w &= 240
\end{aligned}
$$

Incresasing u will increase Z
Increasing u will decrease w.
Increasing u will decrease x.
BUT increasing u will also increase y
so no problem.

A George Dantzig Anecdote



Jerzy Neyman
Born: April 16, 1894 in Bendery, Bessarabia, Russian Empire
Died: August 5, 1981 in Oakland, California