The Simplex Method III

Class 7

February 27, 2023

Announcements and Homework

Notes on Assignment 2

Assignment 3: Due Next Monday

The Linear Programming Problem

$$\text{Maximize } Z = \mathbf{c} \cdot \mathbf{x}$$

subject to

$$A\mathbf{x} \leq \mathbf{b},$$

$$\mathbf{x} \geq \mathbf{0}$$

Linear Programming

A is $m \times n$ matrix of constants and $\mathbf{b}$ is $n \times 1$ vector.

Constraint Set $S = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$

Example: Fromage Cheese Company Problem
$S = \{(x, y) : 30x + 12y \leq 6000, 10x + 8y \leq 2600, 4x + 8y \leq 2000, x \geq 0, y \geq 0\}$

$$A = \begin{bmatrix} 30 & 12 \\ 10 & 8 \\ 4 & 8 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 6000 \\ 2600 \\ 2000 \end{bmatrix}$$

Using Simplex Method
To Solve
Fromage Cheese Company Problem

Maximize $Z = 4.5x + 4y$

subject to the constraints :

$$30x + 12y \leq 6000,$$

$$10x + 8y \leq 2600,$$

$$4x + 8y \leq 2000.$$

$$x \geq 0, \; y \geq 0$$

STEP 1: Introduce slack variables to convert inequalities into equations.

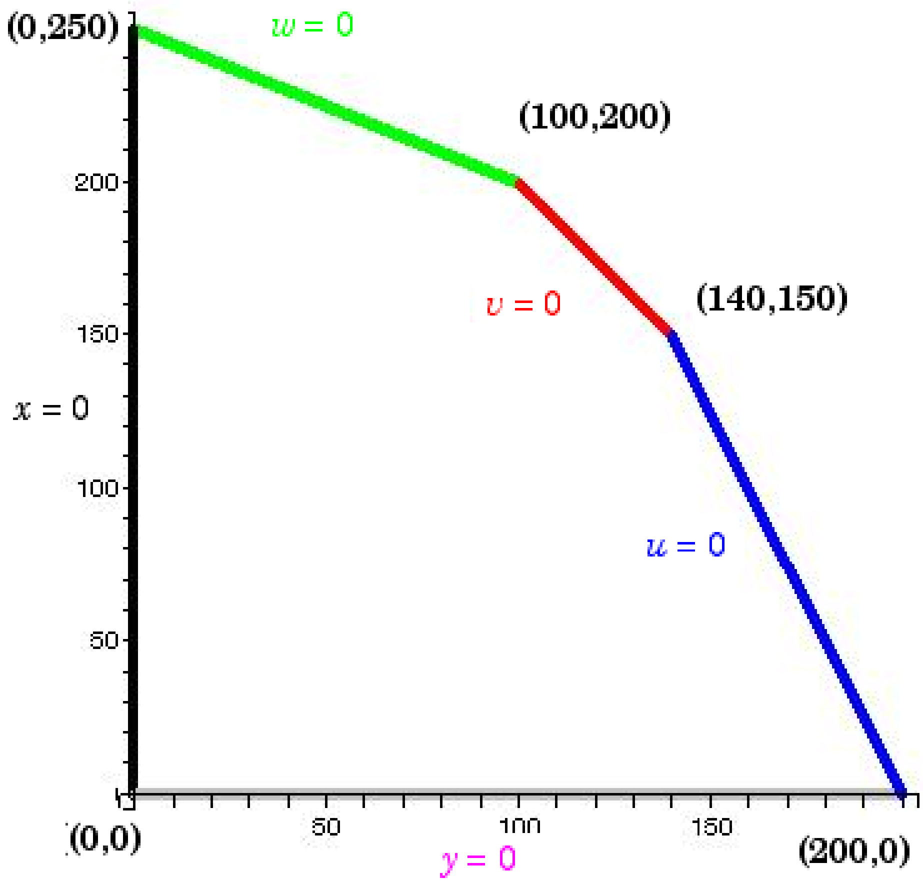
Find nonnegative numbers x, y, u, v, w such that

$$Z = 4.5x + 4y \text{ is maximized}$$

subject to the constraints :

$$\begin{aligned} 30x + 12y + u &= 6000, \\ 10x + 8y + v &= 2600, \\ 4x + 8y + w &= 2000, \end{aligned}$$

$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0,$


(0,250)
w = 0
(100,200)
v = 0
(140,150)
x = 0
u = 0
(0,0)
y = 0
(200,0)
200
150
100
50
50
100
150

$$\begin{array}{ccccc} x & y & u & v & w \end{array}$$
$$\begin{bmatrix} 30 & 12 & 1 & 0 & 0 \\ 10 & 8 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 1 \end{bmatrix}$$

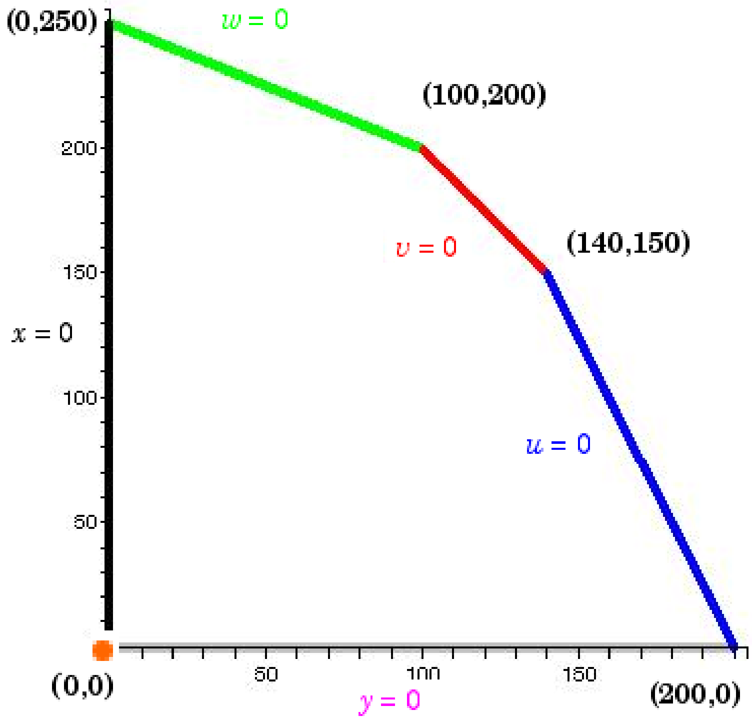
For the cheese example, the solution

$x = 0,\quad y = 0,$
$u = 6000,\quad v = 2600,\ w = 2000.$

is both feasible and basic.

The basic variables are u, v, and w.

Geometrically, this solution is located at the vertex where the two edges $x = 0$ and $y = 0$ intersect.


(0,250)
$w = 0$
(100,200)
200
$v = 0$
(140,150)
150
$x = 0$
100
$u = 0$
50
(0,0)
50
100
150
(200,0)
$y = 0$

This particular solution gives $Z = 0$, which is clearly not optimal. We can increase $Z = 4.5x + 4y$ by increasing either x or y.

One way to go about this is to concentrate on increasing one of the variables.

Since a unit increase in x boosts Z more than a unit increase in y, it is reasonable to begin by making x as large as possible, while keeping $y = 0$. When $y = 0$, our equations can be written

$$u = 6000 - 30x,$$
$$v = 2600 - 10x,$$
$$w = 2000 - 4x.$$

Increase x as much as possible until we drive one of the current basic variables to 0.

$u = 0$ when $30x = 6000$; that is, $x = 6000/30 = 200$

$v = 0$ when $10x = 2600$; that is, $x = 2600/10 = 260$

$w = 0$ when $4x = 2000$; that is, $x = 2000/4 = 500$

The Fromage Cheese Company problem can be formulated as:

Find nonnegative values of x, y, u, v, w such that:

$$\begin{aligned} 30x + 12y + u &= 6000, \\ 10x + 8y + v &= 2600, \\ 4x + 8y + w &= 2000. \\ -4.5x - 4y + Z &= 0 \end{aligned}$$

and so that Z is as large as possible.
(We have represented the objective function as an equation)

We'll put the equation for the objective function first:

$$-4.5x - 4y + Z = 0$$
$$30x + 12y + u = 6000,$$
$$10x + 8y + v = 2600,$$
$$4x + 8y + w = 2000.$$

We write the matrix of coefficients) in a special form, called the *extended simplex tableau* (Tableau 5.1).

Tableau 5.1

	Z	x	y	u	v	w	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000
		↑					

x will enter the basis

Tableau 5.2

$\frac{6000}{30} = 200$

$\frac{2600}{10} = 260$

$\frac{2000}{4} = 500$

	Z	x	y	u	v	w	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	**[30]**	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000
		↑					

x will enter the basis
u will leave the basis

Divide u-row by 30:

Tableau 5.2.1

	Z	x	y	u	v	w	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	**1**	2/5	1/30	0	0	200
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000
		↑					

Subtract (-4.5) *u-row from Z-row
Subtract (10) * u-row from v-row
Subtract (4) * u-row from w-row

Tableau 5.3

	Z	x	y	u	v	w	
Z	**1**	**0**	**-11/5**	**3/20**	**0**	**0**	**900**
x	0	1	2/5	1/30	0	0	200
v	0	0	4	- 1/3	1	0	600
w	0	0	32/5	- 2/15	0	1	1200
			↑				

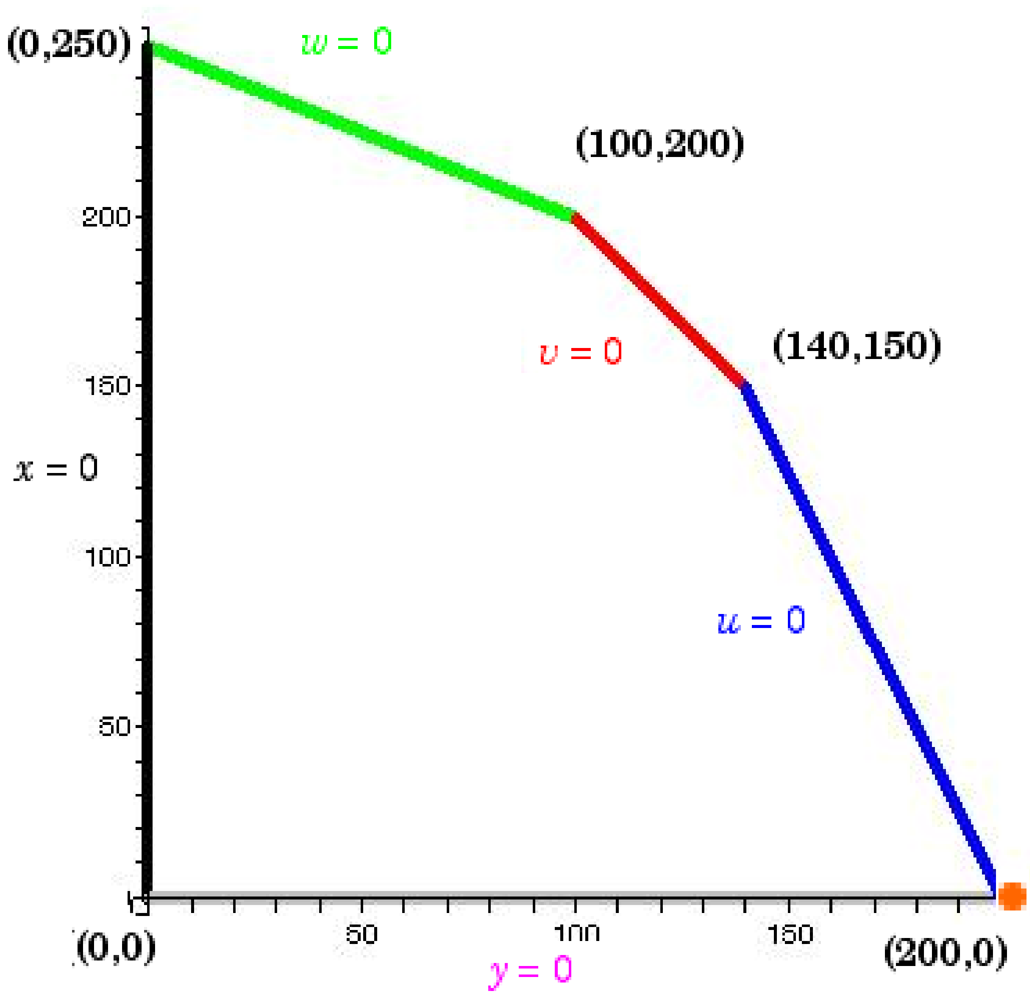


Tableau 5.4

$\frac{200}{2/5} = 500$

$\frac{600}{4} = 150$

$\frac{1200}{32/5} = 187\frac{1}{2}$

	Z	x	y	u	v	w	
Z	**1**	**0**	**-11/5**	**3/20**	**0**	**0**	**900**
x	0	1	2/5	1/30	0	0	200
v	0	0	**[4]**	- 1/3	1	0	600
w	0	0	32/5	- 2/15	0	1	1200
			↑				

y will enter the basis
v will leave the basis

Divide *v*-row by 4
Subtract (-11/5)* new *v*-row from *Z*-row
Subtract (2/5)* new *v*-row from *x*-row
Subtract (32/5)* new *v*-row from *w*-row

Result is

Tableau 5.5.0

	Z	x	y	u	v	w	
Z	**1**	**0**	**0**	**-1/30**	**11/20**	**0**	**1230**
x	0	1	0	1/15	-1/10	0	140
y	0	0	1	- 1/12	1/4	0	150
w	0	0	0	2/5	-8/3	1	240

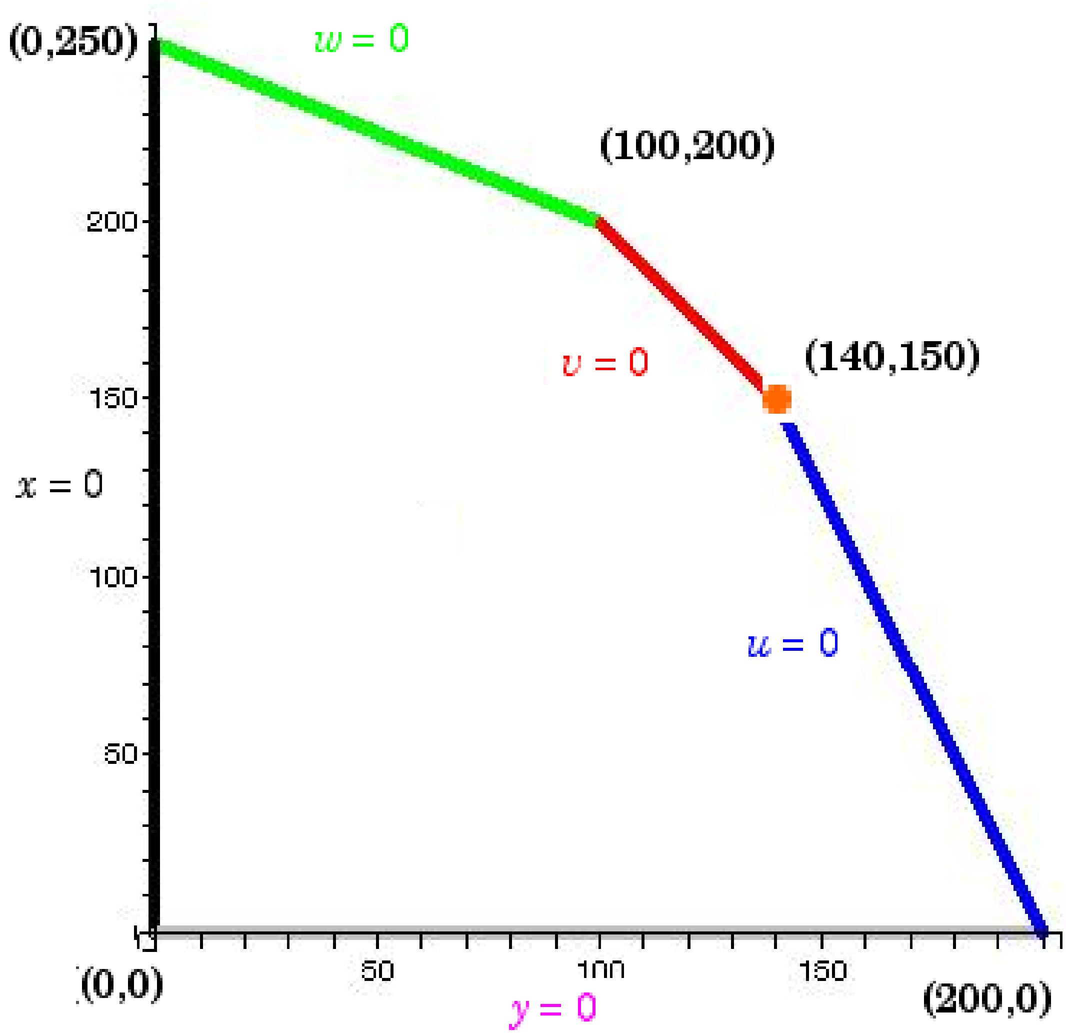


Tableau 5.5

	Z	x	y	u	V	w	
Z	**1**	**0**	**0**	**-1/30**	**11/20**	**0**	**1230**
x	0	1	0	1/15	-1/10	0	140
y	0	0	1	- 1/12	1/4	0	150
w	0	0	0	[2/5]	-8/3	1	240
				↑			

$\frac{140}{1/15} = 2100$

$\frac{240}{2/5} = 600$

u will enter the basis
w will leave the basis

Divide *w*-row by ______

Subtract (-___)* new *w*-row from *Z*-row

Subtract (___)* new *w*-row from *x*-row

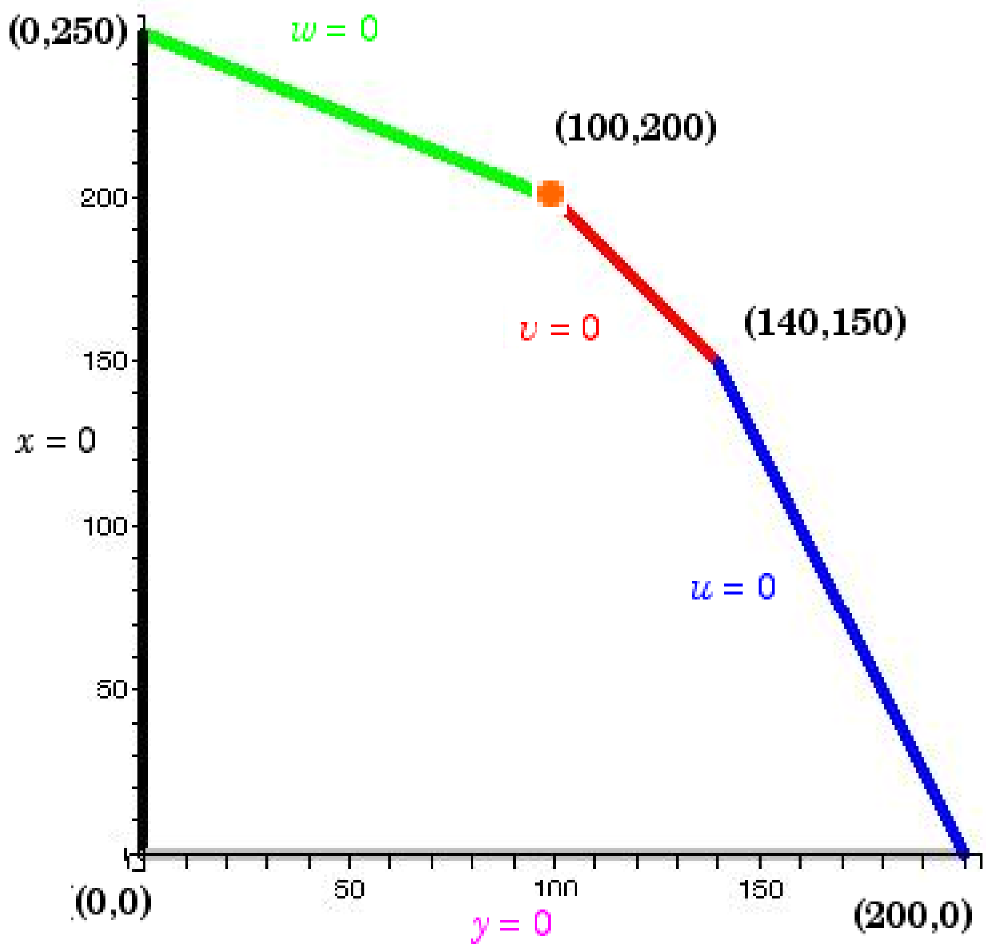
Subtract (___)* new *v*-row from *y*-row

Tableau 5.6

	Z	x	y	u	v	w	
Z	**1**	**0**	**0**	**0**	**5/12**	**1/12**	**1250**
x	0	1	0	0	1/6	-1/6	100
y	0	0	1	0	-1/12	5/24	200
u	0	0	0	1	-4	5/2	600

The new basic feasible solution is

$x = 100$, $y = 200$, $u = 600$, $v = w = 0$, and $Z = 1250$.



The Simplex Method
In a Nutshell

Convert Inequalities to Equations Using Slack Variables.

Create Tableau of Coefficients of the Equations.

Step 1

Start with a feasible tableau: a basic solution that is also non-negative.

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Start with a feasible tableau: a basic solution that is also non-negative.

Test For Optimality:

If the objective function row of the tableau has no negative entries, the solution is optimal.

Step 2

If there is at least one negative entry, choose as the pivot column the one with the largest negative entry in the objective function row, ignoring the last column.

Step 3

For each row except the objective function row, compute θ ratios for all positive entries in the pivot column

$$\theta = \frac{\text{entry in last column}}{\text{entry in pivot column}}$$

θ ratios must all be non-negative.

Step 4

Choose as the pivot row the one with the smallest θ ratio. If there is a tie, decide arbitrarily.

Step 5

Carry out a pivot operation.

Make Pivot Element $= 1$.

Make Other Elements in Pivot Column $= 0$.

Step 6

Repeat steps 2 through 5 until no new pivot column can be found.
A calculation with the simplex method terminates when either no pivot column or pivot row can be found.
If there is no pivot column (no negative entry in the objective function row), then the current solution is optimal.
If there is a pivot column but all its entries are either zero or negative, then one can show that the problem has an unbounded solution, i.e., the objective function can be made arbitrarily large.

GOAL: FIND A SOLUTION ALGORITHM

which will

1. Find an optimal feasible solution, if it exists, in an efficient manner.
2. Tell us if the objective function is unbounded.
3. Tell us if the constraint set is empty.

$$
\begin{aligned}
-4.5x - 4y + Z \quad &= 0 \\
30x + 12y + u \quad &= 6000 \\
10x + 8y + v \quad &= 2600 \\
4x + 8y + w \quad &= 2000
\end{aligned}
$$

	Z	x	y	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

The tableau after several iterations:

	Z	x	y	u	v	w	
Z	1	0	0	$-1/30$	$11/20$	0	1230
x	0	1	0	$1/15$	$-1/10$	0	140
y	0	0	1	$-1/12$	$1/4$	0	150
w	0	0	0	$2/5$	$-8/3$	1	240

Current values: $x = 140, y = 150, w = 240, u = 0, v = 0$
Basic variables: x, y, w
Nonbasic variables: u, v
u will enter the basis.

$$\begin{aligned} Z\text{-}\mathbf{1/30\ u} + 11/20v &= 1230 \\ x + \mathbf{1/15\ u} - 1/10v &= 140 \\ y\text{-}\mathbf{1/12\ u} + 1/4v &= 150 \\ \mathbf{2/5\ u} - 8/3v + w &= 240 \end{aligned}$$

Current values: $x = 140, y = 150, w = 240, u = 0, v = 0$
Basic variables: x, y, w
Nonbasic variables: u, v
u will enter the basis.

$$
\begin{aligned}
Z\text{-}\mathbf{1/30\ u} + 11/20v &= 1230 \\
x + \mathbf{1/15\ u} - 1/10v &= 140 \\
y\text{-}\mathbf{1/12\ u} + 1/4v &= 150 \\
\mathbf{2/5\ u} - 8/3v + w &= 240
\end{aligned}
$$

Increasing u will increase Z
Increasing u will decrease w.
Increasing u will decrease x.
BUT increasing u will also increase y
so no problem.

How Simplex Method Recognizes Unbounded Problem

Suppose tableau looks like

	Z	x	y	u	v	w	
Z	1	0	0	$-1/30$	$11/20$	0	1230
x	0	1	0	-0	$-1/10$	0	140
y	0	0	1	$-1/12$	$1/4$	0	150
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How Simplex Method Recognizes Unbounded Problem

Suppose tableau looks like

	Z	x	y	u	v	w	
Z	1	0	0	$-1/30$	$11/20$	0	1230
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y	0	0	1	$-1/12$	$1/4$	0	150
w	0	0	0	$-2/5$	$-8/3$	1	240

If there is a pivot column but all its entries are either zero or negative, then the problem has an unbounded solution, i.e., the objective function can be made arbitrarily large