Introduction to the Simplex Method

Class 6

February 24, 2023

Homework and Announcements

Assignment 2: Due On Monday

For Problem 4 and Hillier-Lieberman 3.4-17a: Only need to **formulate** the problem, not solve it!

Other Hillier-Lieberman problems involve Graphical approach:
Straight lines and convex sets in the plane

On Assignment 2

Problem 1: Let $N =$ number of possible interviews, say 3. Then

Expected Value of $N = p_1 \times 1 + p_2 \times 2 + p_3 \times 3$,

where

p_1 is probability of having **exactly** 1 interview,

p_2 is probability of having **exactly** 2 interviews, and

p_3 is probability of having **exactly** 3 interviews

if you follow Optimal Strategy.

Note: $p_1 + p_2 + p_3 = 1$.

Handout

Linear Programming Part III

The Linear Programming Problem

Maximize $Z = \mathbf{c} \cdot \mathbf{x}$

subject to

$A\mathbf{x} \leq \mathbf{b},$

$\mathbf{x} \geq \mathbf{0}$

Linear Programming

A is $m \times n$ matrix of constants and $\mathbf{b}$ is $n \times 1$ vector.

Constraint Set $S = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$

Example: Fromage Cheese Company Problem
$S = \{(x, y) : 30x + 12y \leq 6000, 10x + 8y \leq 2600, 4x + 8y \leq 2000, x \geq 0, y \geq 0\}$

$$A = \begin{bmatrix} 30 & 12 \\ 10 & 8 \\ 4 & 8 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 6000 \\ 2600 \\ 2000 \end{bmatrix}$$

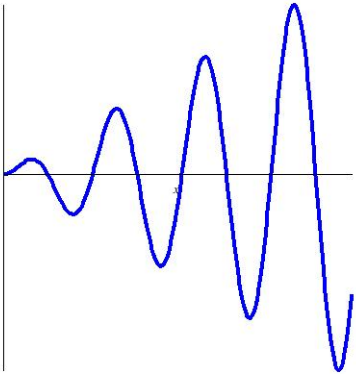
The Simplex Method

George Bernard Dantzig
(November 8, 1914 - May 13, 2005)

A Possible Problem with the Simplex Method

Simplex Method Finds **LOCAL** Maximum Only

Local Maximum Is Not Necessarily Global Maximum



A Possible Problem with the Simplex Method

Simplex Method Finds **LOCAL** Maximum Only

A Possible Problem with the Simplex Method

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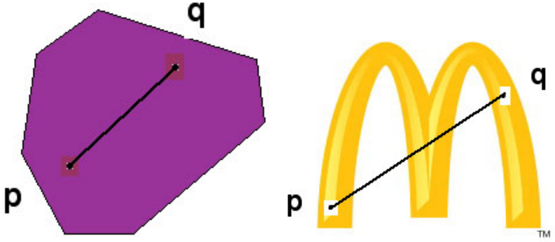
A Possible Problem with the Simplex Method

Simplex Method Finds **LOCAL** Maximum Only

Could A be local max while D is the global max?

Convex Sets

A set K in R^n is **convex** if the entire line segment connecting any pair of points in the set lies entirely in the set.



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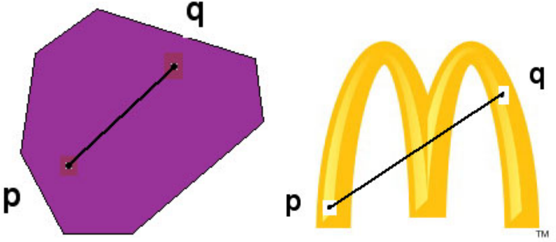


K is convex if and only if $\{t\mathbf{q} + (1-t)\mathbf{p}\}$ is in K for every $\mathbf{p}$, $\mathbf{q}$ in K and all $t, 0 \leq t \leq 1$.

Intersection of Convex Sets is Convex

Each LP constraint defines a convex set

Theorem

Let $\mathbf{a}$ *be any vector in* R^n *and let* b *be any real number. Then* $H = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} \leq b\}$ *is convex.*

A similar proof shows that each of these sets is also convex:

$$\{\mathbf{x} : \mathbf{a}^T\mathbf{x} \leq b\}$$
$$\{\mathbf{x} : \mathbf{a}^T\mathbf{x} = b\}$$
$$\{\mathbf{x} : \mathbf{a}^T\mathbf{x} \geq b\}$$

Major Conclusion: The Feasibility Set of an LP Problem is Convex

Theorem

For an LP Problem. any local maximum is a global maximum.

Proof.

Suppose $\mathbf{p}$ is a local maximum and $\mathbf{q}$ is a global maximum with $f(\mathbf{q}) > f(\mathbf{p})$ Let $\mathbf{r}$ be any point of the form

$$\mathbf{r} = \lambda\mathbf{q} + (1-\lambda)\mathbf{p} \text{ with } 0 < \lambda < 1$$

Then

$$\begin{aligned} f(\mathbf{r}) &= \lambda f(\mathbf{q}) + (1-\lambda)f(\mathbf{p}) \\ &> \lambda f(\mathbf{p}) + (1-\lambda)f(\mathbf{p}) \\ &= f(\mathbf{p}) \end{aligned}$$

so $f(\mathbf{r}) > f(\mathbf{p})$
Thus $\mathbf{p}$ is not a local maximum

□

GOAL: FIND A SOLUTION ALGORITHM

which will

1. Find an optimal feasible solution, if it exists, in an efficient manner.
2. Tell us if the objective function is unbounded.
3. Tell us if the constraint set is empty.

Using Simplex Method
To Solve
Fromage Cheese Company Problem

Maximize $Z = 4.5x + 4y$

subject to the constraints :

$30x + 12y \leq 6000,$

$10x + 8y \leq 2600,$

$4x + 8y \leq 2000.$

$x \geq 0, \; y \geq 0$

STEP 1: Introduce slack variables to convert inequalities into equations.

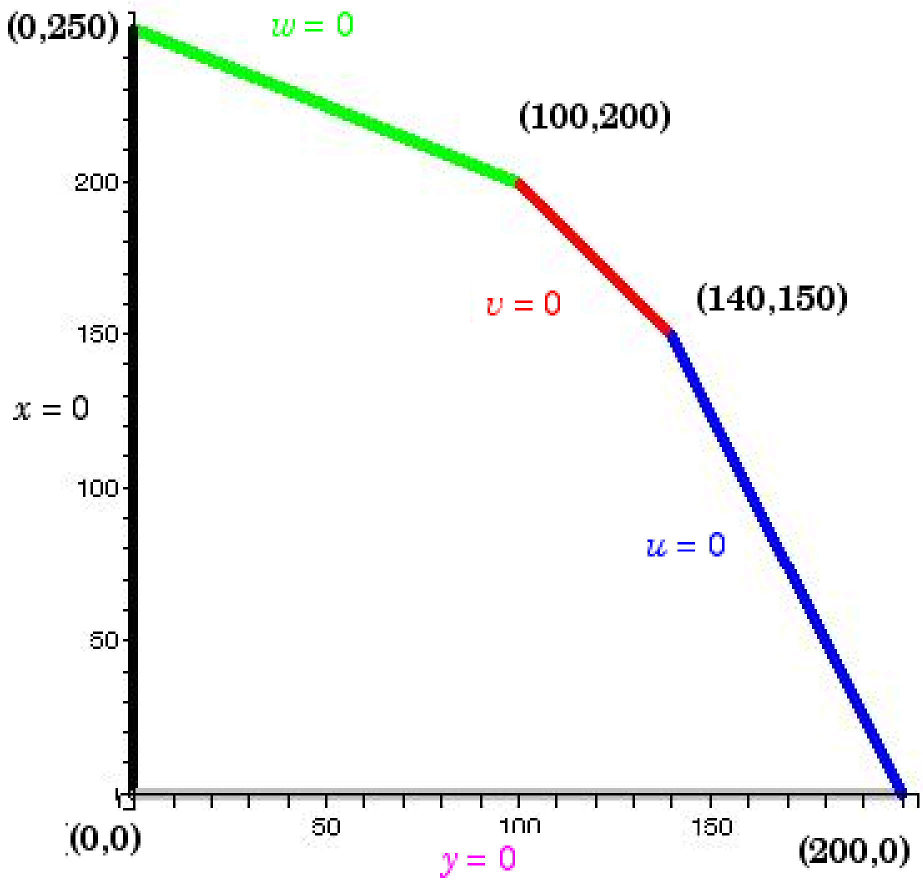
Find nonnegative numbers x, y, u, v, w such that

$$Z = 4.5x + 4y \text{ is maximized}$$

subject to the constraints :

$$\begin{aligned} 30x + 12y + u &= 6000, \\ 10x + 8y + v &= 2600, \\ 4x + 8y + w &= 2000, \end{aligned}$$

$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0,$


(0,250)
w = 0
(100,200)
200
v = 0
(140,150)
150
x = 0
100
u = 0
50
(0,0)
50
100
150
(200,0)
y = 0

$$\begin{array}{ccccc} x & y & u & v & w \end{array}$$

$$\begin{bmatrix} 30 & 12 & 1 & 0 & 0 \\ 10 & 8 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 1 \end{bmatrix}$$

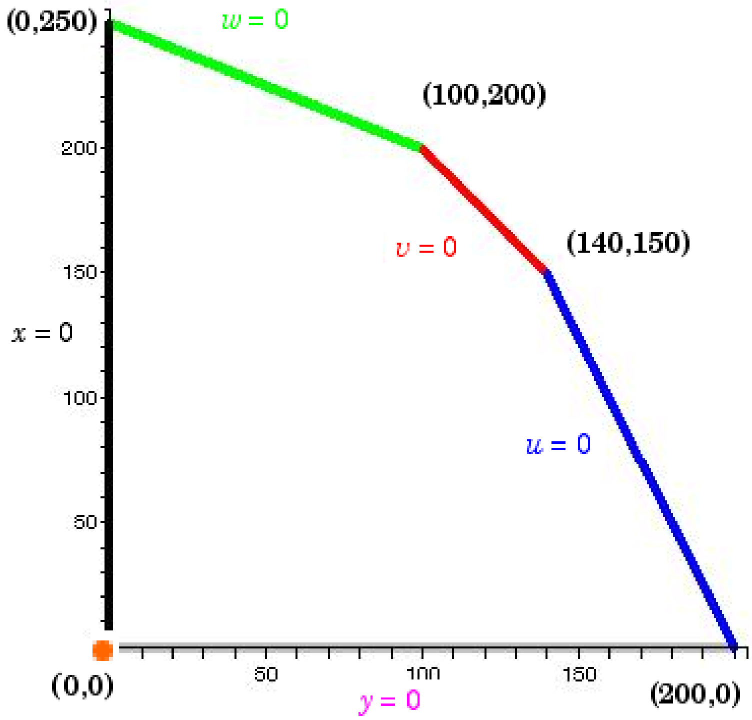
For the cheese example, the solution

$x = 0, \quad y = 0,$
$u = 6000, \quad v = 2600, \quad w = 2000.$

is both feasible and basic.

The basic variables are u, v, and w.

Geometrically, this solution is located at the vertex where the two edges $x = 0$ and $y = 0$ intersect.


(0,250)
w = 0
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x = 0
100
u = 0
50
(0,0)
50
100
150
(200,0)
y = 0

This particular solution gives $Z = 0$, which is clearly not optimal. We can increase $Z = 4.5x + 4y$ by increasing either x or y.

One way to go about this is to concentrate on increasing one of the variables.

Since a unit increase in x boosts Z more than a unit increase in y, it is reasonable to begin by making x as large as possible, while keeping $y = 0$. When $y = 0$, our equations can be written

$$u = 6000 - 30x,$$
$$v = 2600 - 10x,$$
$$w = 2000 - 4x.$$

Increase x as much as possible until we drive one of the current basic variables to 0.

$u = 0$ when $30x = 6000$; that is, $x = 6000/30 = 200$

$v = 0$ when $10x = 2600$; that is, $x = 2600/10 = 260$

$w = 0$ when $4x = 2000$; that is, $x = 2000/4 = 500$

The Fromage Cheese Company problem can be formulated as:

Find nonnegative values of x, y, u, v, w such that:

$$30x + 12y + u = 6000,$$
$$10x + 8y + v = 2600,$$
$$4x + 8y + w = 2000.$$
$$-4.5x - 4y + Z = 0$$

and so that Z is as large as possible.
(We have represented the objective function as an equation)

We'll put the equation for the objective function first:

$$-4.5x - 4y + Z = 0$$
$$30x + 12y + u = 6000,$$
$$10x + 8y + v = 2600,$$
$$4x + 8y + w = 2000.$$

We write the matrix of coefficients) in a special form, called the *extended simplex tableau* (Tableau 5.1).

Tableau 5.1

	Z	x	y	u	v	w	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000
		↑					

x will enter the basis

Tableau 5.2

$\frac{6000}{30} = 200$

$\frac{2600}{10} = 260$

$\frac{2000}{4} = 500$

	Z	x	y	u	v	w	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	**[30]**	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000
		↑					

x will enter the basis
u will leave the basis

Divide u-row by 30:

Tableau 5.2.1

	Z	x	y	u	v	w	
Z	**1**	**-4.5**	**-4**	**0**	**0**	**0**	**0**
u	0	**1**	2/5	1/30	0	0	200
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000
		↑					

Subtract (-4.5) *u-row from Z-row
Subtract (10) * u-row from v-row
Subtract (4) * u-row from w-row

Tableau 5.3

	Z	x	y	u	v	w	
Z	**1**	**0**	**-11/5**	**3/20**	**0**	**0**	**900**
x	0	1	2/5	1/30	0	0	200
v	0	0	4	- 1/3	1	0	600
w	0	0	32/5	- 2/15	0	1	1200
			↑				

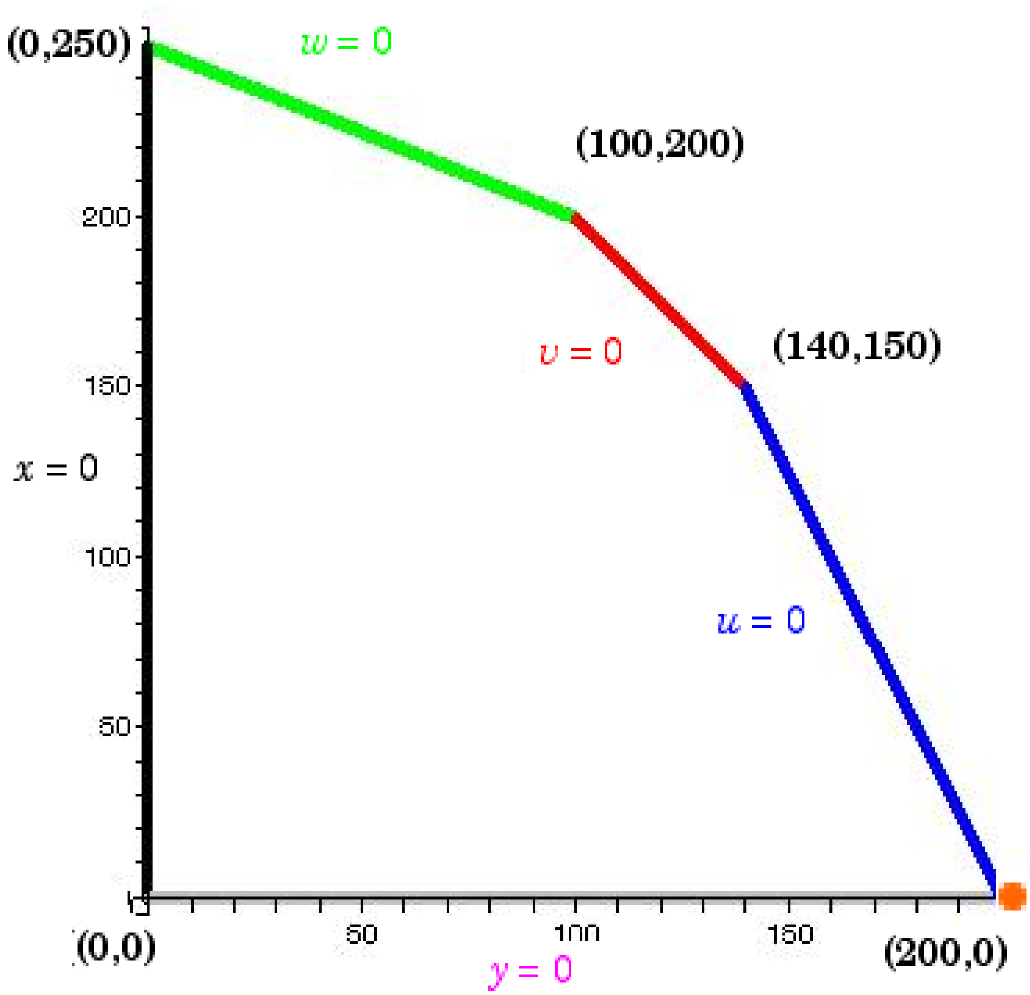

Tableau 5.4

$\frac{200}{2/5} = 500$

$\frac{600}{4} = 150$

$\frac{1200}{32/5} = 187\frac{1}{2}$

	Z	x	y	u	v	w	
Z	**1**	**0**	**-11/5**	**3/20**	**0**	**0**	**900**
x	0	1	2/5	1/30	0	0	200
v	0	0	**[4]**	- 1/3	1	0	600
w	0	0	32/5	- 2/15	0	1	1200
			↑				

y will enter the basis
v will leave the basis

Divide *v*-row by 4
Subtract (-11/5)* new *v*-row from *Z*-row
Subtract (2/5)* new *v*-row from *x*-row
Subtract (32/5)* new *v*-row from *w*-row

Result is

Tableau 5.5.0

	Z	x	y	u	v	w	
Z	**1**	**0**	**0**	**-1/30**	**11/20**	**0**	**1230**
x	0	1	0	1/15	-1/10	0	140
y	0	0	1	- 1/12	1/4	0	150
w	0	0	0	2/5	-8/3	1	240

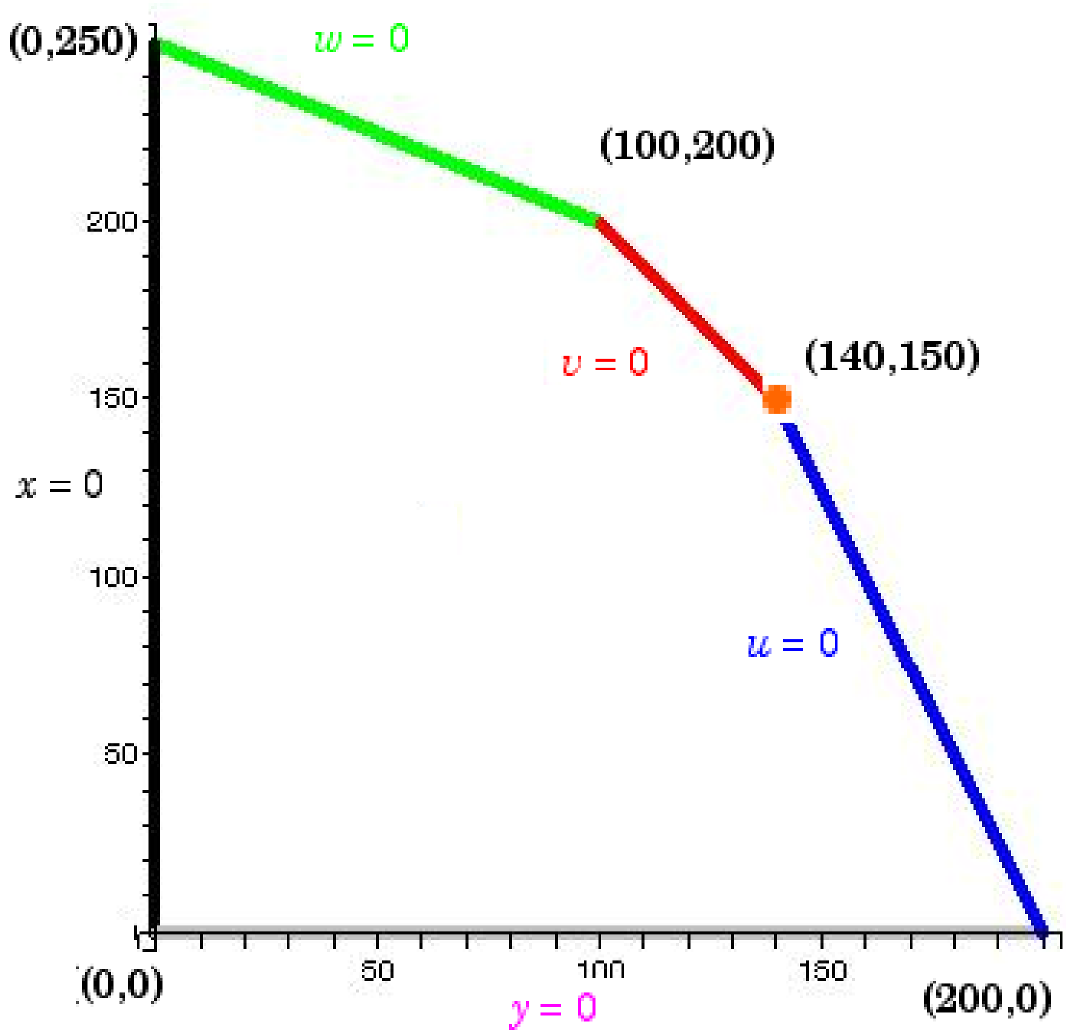

Tableau 5.5

		Z	x	y	u	V	w	
	Z	**1**	**0**	**0**	**-1/30**	**11/20**	**0**	**1230**
$\frac{140}{1/15} = 2100$	x	0	1	0	1/15	-1/10	0	140
	y	0	0	1	- 1/12	1/4	0	150
$\frac{240}{2/5} = 600$	w	0	0	0	[2/5]	-8/3	1	240
					↑			

u will enter the basis
w will leave the basis

Divide *w*-row by ______

Subtract (-___)* new *w*-row from *Z*-row

Subtract (___)* new *w*-row from *x*-row

Subtract (___)* new *v*-row from *y*-row

Tableau 5.6

	Z	x	y	u	v	w	
Z	**1**	**0**	**0**	**0**	**5/12**	**1/12**	**1250**
x	0	1	0	0	1/6	-1/6	100
y	0	0	1	0	-1/12	5/24	200
u	0	0	0	1	-4	5/2	600

The new basic feasible solution is

$x = 100$, $y = 200$, $u = 600$, $v = w = 0$, and $Z = 1250$.