Constraint Sets for Linear Programming

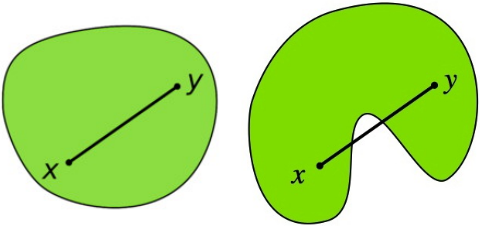
Class 5

February 22, 2023

Our Hero

Reading

Linear Programming II Notes: Convex Sets (on line)



Review: Behavior of Linear Objective Functions

Suppose f is a nonconstant linear function defined on a set S.

If f has a maximum on S, then it must occur on the boundary of S.

If S has a polygonal boundary, then the maximum value will occur at a vertex.

THE MAXIMUM VALUE OF A LINEAR FUNCTION ON A POLYGONAL SET, IF IT EXISTS, ALWAYS OCCURS AT A VERTEX

Level Sets

Let $S \subset R^n$ and $f : S \to R$ be a real-valued function defined on S

Then a **level set** for f is a set $L_k = \{\mathbf{x} : f(\mathbf{x}) = k\}$ for some constant k.

For a **linear** function, level set $= \{\mathbf{x} : \mathbf{c}^T\mathbf{x} = k\}$
$= \{\mathbf{x} : c_1x_1 + c_2x_2 + c_3x_3 + ... + c_nx_n = k\}$

$n = 2$: line in the plane
$n = 3$: plane in 3-space
$n = 4$ flat 3-space in 4-space

In general, a level set is an $n - 1$ hyperplane in R^n.
These level sets form a collection of parallel hyperplanes filling up R^n.
Moreover, the vector $\mathbf{c}$ is perpendicular to each of the these hyperplanes and points in the direction of increasing values of f.

Important Types of Constrained Linear Objective Function Problems

1. **Integer Programming**

$$Z^n = \{\mathbf{x} \in R^n : \text{all components of } \mathbf{x} \text{ are integers } \}$$
$$S \subset Z^N$$

Examples: Assignment Problem, Sudoku

2. **Combinatorial Programming**

S is the set of all permutations of the first n positive integers

Example: Traveling Salesperson's Problem

3. **Linear Programming**

TSP: Traveling Salesperson's Problem

You must visit n cities, denoted 1,2,3,..., n in some order.
There is a certain cost c_{ij} in traveling from city i to city j.
Problem: Choose the order that minimizes total cost.

Order $=$ (1,2,3,4) has Cost $= c_{12} + c_{23} + c_{34}$
Order $=$ (3,1,2,4) has Cost $= c_{31} + c_{12} + c_{24}$

TSP of 50 state capitols: $n = 50$
Number of different orderings $= 50! \approx 3.04 \times 10^{64}$
Brute Force Attack? Check one billion per second: 9.6×10^{47} years.
Age of Universe: 2×10^{10} years.

Linear Programming

A is $m \times n$ matrix of constants and $\mathbf{b}$ is $n \times 1$ vector.

Constraint Set $S = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$

Example: Fromage Cheese Company Problem
$S = \{(x, y) : 30x + 12y \leq 6000, 10x + 8y \leq 2600, 4x + 8y \leq 2000, x \geq 0, y \geq 0\}$

$$A = \begin{bmatrix} 30 & 12 \\ 10 & 8 \\ 4 & 8 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 6000 \\ 2600 \\ 2000 \end{bmatrix}$$

Level Sets for Fromage Cheese Company Problem

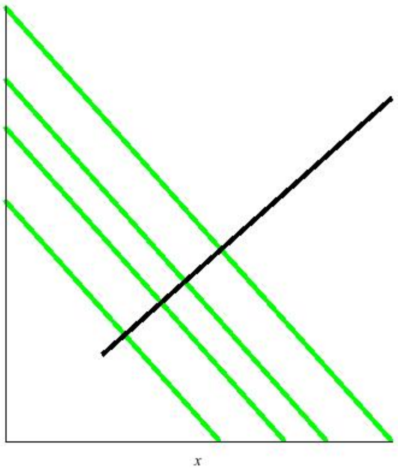
$f := R^2 \to R$ by $f(x, y) = 4.5x + 4y$

Level set with level k is set of solutions of $4.5x + 4y = k$

This is a line containing

$(0, \frac{k}{4})$ and $(\frac{2k}{9}, 0)$

Orthogonality of **c** to Level Sets



The Fromage Cheese Company Problem

Assortment	Cheddar	Swiss	Brie	Price	Number of Packages
Fancy	30	10	4	\$4.50	x
Deluxe	12	8	8	\$4.00	y
Available	6000	2600	2000		

Note: All quantities of cheeses are in ounces

Question: How many packages of each mixture should we make that will maximize revenue but not exceed our cheese supplies?

$$\begin{gathered}\text{Problem: Maximize } M = 4.5x + 4y\\ \text{subject to constraints}\\ 30x + 12y \leq 6000 \text{ (Cheddar)}\\ 10x + 8y \leq 2600 \text{ (Swiss)}\\ 4x + 8y \leq 2000 \text{ (Brie)}\\ x, y \geq 0\end{gathered}$$

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$$A = \begin{bmatrix} 30 & 12 \\ 10 & 8 \\ 4 & 8 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 6000 \\ 2600 \\ 2000 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

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Fromage Problem: Maximize $4.5x + 4y$

Boundary Constraint	x-intercept	y-intercept	color
$30x + 12y = 6000$	(200,0)	(0,500)	blue
$10x + 8y = 2600$	(260,0)	(0,325)	red
$4x + 8y = 2000$	(500,0)	(0,250)	green

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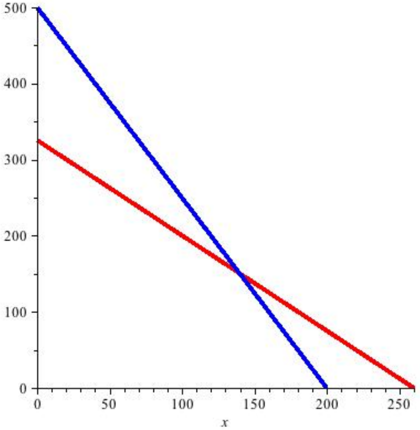

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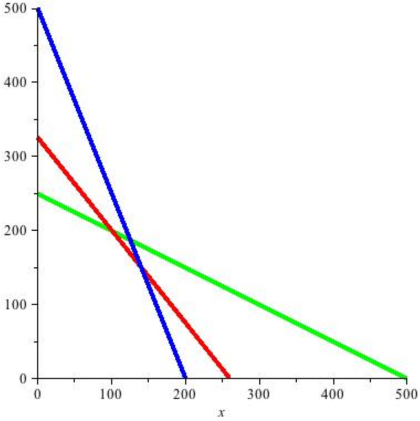
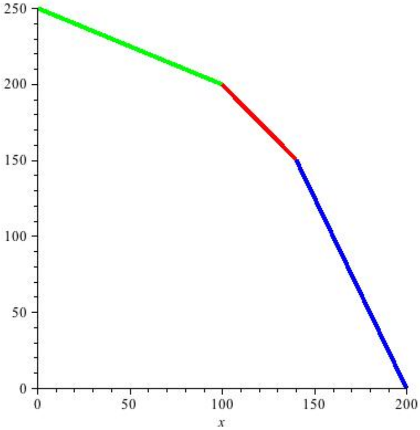
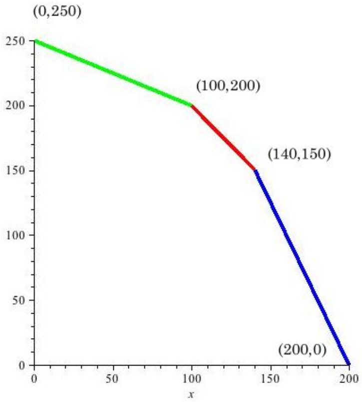


The Constraint Set for the Fromage Cheese Company Problem

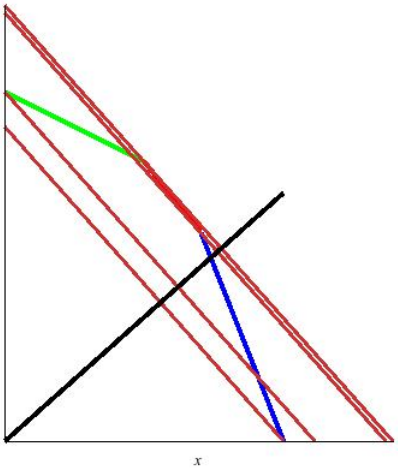


Vertex	(0,0)	(200,0)	(140,150)	(100,200)	(0,250)
Revenue	0	900	1230	1250	1000



Moving along **c**, starting at (0,0), we pass

(200,0)	\$900
(0,250)	\$1000
(140,150)	\$1230
(100,200)	\$1250

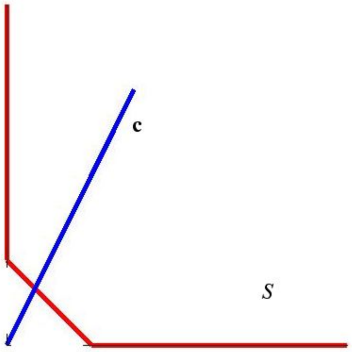


GOAL: FIND A SOLUTION ALGORITHM

which will

1. Find an optimal feasible solution, if it exists, in an efficient manner.
2. Tell us if the objective function is unbounded.
3. Tell us if the constraint set is empty.

An LP problem need not have an optimal feasible solution.
The objective function could be unbounded.



THE SIMPLEX METHOD IS A SOLUTION ALGORITHM

which

1. Finds an optimal feasible solution, if it exists, in an efficient manner.
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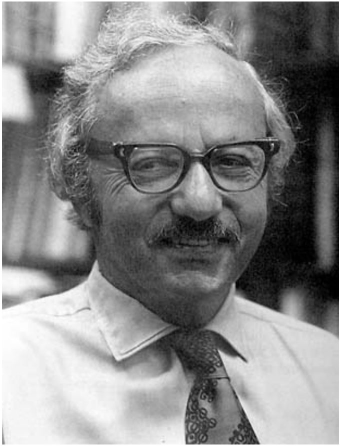
The Simplex Method



George Bernard Dantzig
(November 8, 1914 - May 13, 2005)

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Constraint Set $S = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$

Eaxh row of $A\mathbf{x} \leq \mathbf{b}$ is an inequality of the form
$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \leq b_i$
which is the equation of a closed half-space in R^n.

S is the intersection of a finite number of closed half-spaces.

A Closed Half-Space is Convex

A set S is **convex** means that whenever P and Q are points in S, then so is every point on the line segment between P and Q.

Let H be a half-space. Then $H = \{\mathbf{x} : \mathbf{a}^T \cdot \mathbf{x} \leq b\}$
for some vector **a** and some constant b.
Suppose **y** and **z** belong to H
so $\mathbf{a}^T \cdot \mathbf{y} \leq b$ and $\mathbf{a}^T \cdot \mathbf{z} \leq b$
Let **x** be an any point on the line segment between **y** and **z**
Then $\mathbf{x} = t\mathbf{y} + (1-t)\mathbf{z}$ for some t between 0 and 1.
We need to show $\mathbf{a}^T \cdot \mathbf{x} \leq b$

$$\mathbf{a}^T \cdot \mathbf{x} = \mathbf{a}^T \cdot (t\mathbf{y} + (1-t)\mathbf{z}) = \mathbf{a}^T \cdot (t\mathbf{y}) + \mathbf{a}^T \cdot ((1-t)\mathbf{z})$$
$$= t(\mathbf{a}^T \cdot \mathbf{y}) + (1-t)(\mathbf{a}^T \cdot \mathbf{z}) \leq tb + (1-t)b = b.$$

Theorem: The intersection of any collection of convex sets is convex.

Proof: Let P and Q belong to the intersection and let R be any point on the line segment between P and Q.

Since P and Q belong to the intersection, they belong to each of the convex sets in the collection.

But each set in the collection is convex.

So R belongs to each set in the collection and hence R belongs to the intersection.

Corollary: The constraint set in a Linear Programming Problem is convex.

Proof: The constraint set is the intersection of a finite collection of half-spaces, each of which is convex.