Introduction to Operations Research

Class 4

February 20, 2023

Homework

Notes on Assignment 1

Assignment 2: Problem 3: Should be able to run *FirstSimulation* directly from the webpage or you can copy *FirstSimulation.html* from Handout Folder to your desktop

General Mathematical Programming Problem

Given

A set $S \subset R^n$ called the **Constraint Set**

and

A real-valued function $f : S \to R^1$ called the **Objective Function**

Want

$$\sup_{\mathbf{x} \in S} f(\mathbf{x})$$

Review: Behavior of Linear Objective Functions

Last Time: informal argument that if **x** is an **interior** point of the constraint set, then we can increase the value of a linear objective function f by moving to a point of the form $\mathbf{x} + \lambda\,\mathbf{c}$ if $\lambda > 0$.

Thus if f has a maximum on S, then it must occur on the boundary of S.

If S is two-dimensional with a polygonal boundary, then the maximum value will occur at a vertex.

A More Formal Approach

Theorem

If a constrained optimization problem with a linear objective function has an optimal feasible solution at some point, then that point is on the boundary of the constraint set

Recall: **norm** of a vector $\mathbf{x} = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$
and

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$$

If $\mathbf{x}_0$ is an interior point of S, then there is a positive number r such that

$$B = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < r\} \subset S$$

B is called the *Open Ball of radius r centered at x_0*.

Geometrically, from $\mathbf{x}_0$, it is possible to look in all directions a positive distance r and see only points of S.
In particular, we can move some positive distance in the direction of $\mathbf{c}$ along a line segment that lies entirely in S.

Even more specifically, the point

$$\mathbf{y} = \mathbf{x}_0 + \frac{r}{2|\mathbf{c}|}\ \mathbf{c}$$

lies inside S for

$$|\mathbf{y} \text{ - } \mathbf{x}_0| = |\frac{r}{2|\mathbf{c}|}\mathbf{c}| = \frac{r}{2|\mathbf{c}|}|\mathbf{c}| = \frac{r}{2} < r$$

Thus $\mathbf{y}$ is a feasible solution.

Proof: No Maximum at an Interior Point

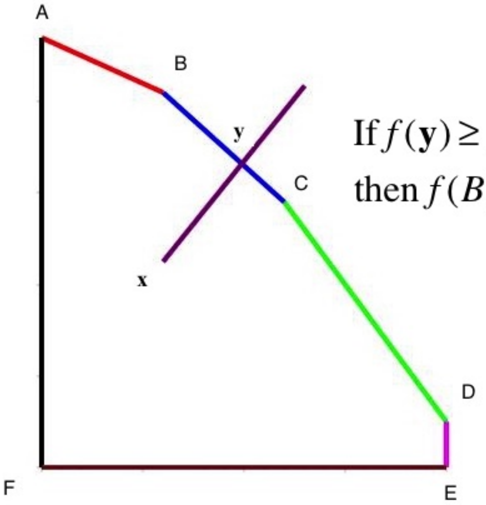
$$
\begin{aligned}
f(\mathbf{y}) &= \mathbf{c}^T y \\
&= \mathbf{c}^T(\mathbf{x}_0 + \frac{r}{2|\mathbf{c}|}\mathbf{c}) \\
&= \mathbf{c}^T\mathbf{x}_0 + \frac{r}{2|\mathbf{c}|}\mathbf{c}^T\mathbf{c} \\
&= \mathbf{c}^T\mathbf{x}_0 + \frac{r}{2|\mathbf{c}|}|\mathbf{c}|^2 \\
&= f(\mathbf{x}_0) + \frac{r}{2}|\mathbf{c}| > f(\mathbf{x}_0)
\end{aligned}
$$

Linear Functions on the Boundary of S

Now suppose $\mathbf{x}$ is on the boundary of S and there is some vector $\mathbf{d}$ so that $\mathbf{x} + t\ \mathbf{d}$ is also contained in the boundary of S for all sufficiently small t; that is, t can range over some interval containing both positive and negative numbers.
Then

$$f(\mathbf{x} + t\mathbf{d}) = \mathbf{c}^T(\mathbf{x} + t\mathbf{d}) = \mathbf{c}^T\mathbf{x} + t\mathbf{c}^T\mathbf{d} = f(\mathbf{x}) + t(\mathbf{c}^T\mathbf{d})$$

$\mathbf{c}^T \cdot \mathbf{d} > 0$: increase f by moving in direction of $\mathbf{d}$ for $t > 0$
$\mathbf{c}^T \cdot \mathbf{d} = 0$: No change in value of f
$\mathbf{c}^T \cdot \mathbf{d} < 0$: increase f by moving in direction of $\mathbf{d}$ for $t < 0$



If $f(\mathbf{y}) \geq f(\mathbf{x})$,

then $f(B)$ or $f(C) \geq f(\mathbf{y})$

THE MAXIMUM VALUE OF A LINEAR FUNCTION ON A POLYGONAL SET, IF IT EXISTS, ALWAYS OCCURS AT A VERTEX

A POLYGONAL SET HAS ONLY FINITELY MANY VERTICES

Our Hero

Level Sets

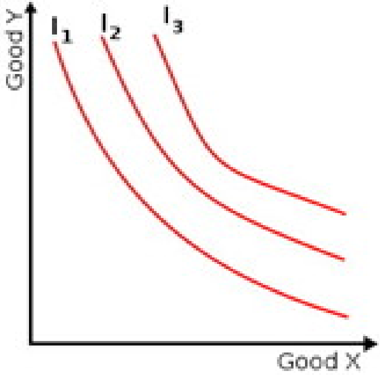
Let $S \subset R^n$ and $f : S \to R$ be a real-valued function defined on S

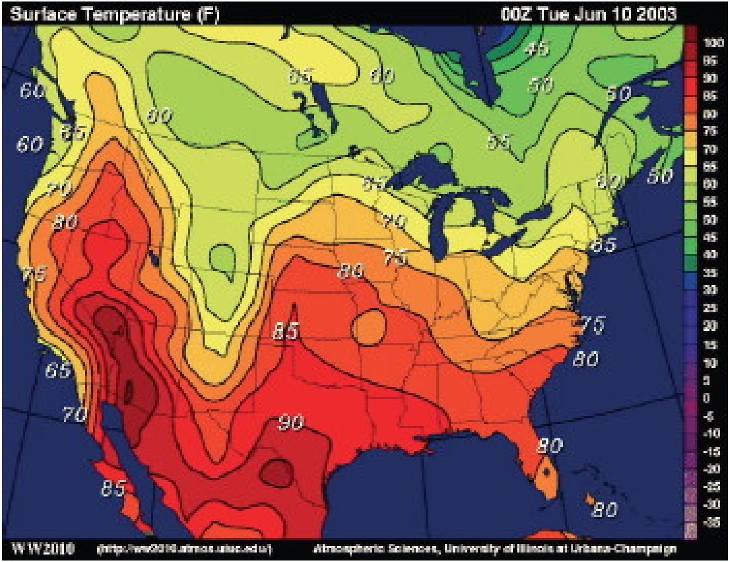
Then a **level set** for f is a set $A = \{\mathbf{x} : f(\mathbf{x}) = k\}$ for some constant k.

Examples:

1. $f : R^2 \to R$ by $f(x, y) = x^2 + y^2$
   Then $\{\mathbf{x} : f(\mathbf{x}) = 1\}$ is the unit circle
2. $f : R^3 \to R$ by $f(x, y, z) = x^2 + y^2 + z^2$
   Then $\{\mathbf{x} : f(\mathbf{x}) = 9\}$ is a sphere of radius 3, center at origin.
3. If f is a temperature, then a level set for f is an isotherm.
4. If f is a utility function, then a level set for f is an indifference curve.

Indifference Curves Are Level Sets




Surface Temperature (F)
00Z Tue Jun 10 2003
WW2010
Atmospheric Sciences, University of Illinois at Urbana-Champaign

Today's Isotherms

Today's Surface Temperatures

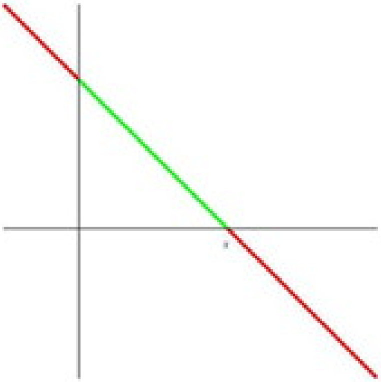
Level Sets for Fromage Cheese Company Problem

$$f := R^2 \to R \text{ by } f(x,y) = 4.5x + 4y$$

Level set with level k is set of solutions of $4.5x + 4y = k$

This is a line containing

$(0, \frac{k}{4})$ and $(\frac{2k}{9}, 0)$

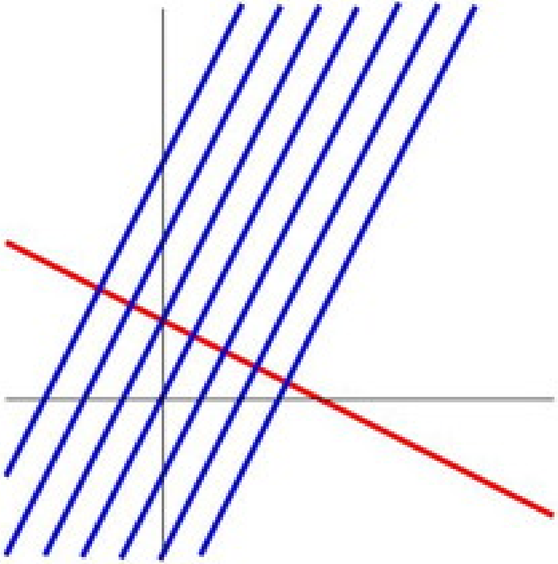


Level Sets for Linear Function

For a linear function, level set $= \{\mathbf{x} : \mathbf{c}^T(x) = k\}$
$= \{\mathbf{x} : c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = k\}$

$n = 2$: line in the plane
$n = 3$: plane in 3-space
$n = 4$ flat 3-space in 4-space

In general, a level set is an $n - 1$ hyperplane in R^n.
These level sets form a collection of parallel hyperplanes filling up R^n. Moreover, the vector $\mathbf{c}$ is perpendicular to each of the these hyperplanes and points in the direction of increasing values of f.



Orthogonality of $\mathbf{c}$ to Level Sets

Verify claim that the vector $\mathbf{c}$ is perpendicular to each of the level sets of f:

Let $L_k = \{\mathbf{x} : \mathbf{c}^T\mathbf{x} = k\}$

Suppose $\mathbf{x}$ and $\mathbf{y}$ are in L_k.

Then examine $\mathbf{v} = \mathbf{x} - \mathbf{y}$

and $\mathbf{c}^T\mathbf{v} = c^T(\mathbf{x} - \mathbf{y}) = c^T\mathbf{x} - c^T\mathbf{y} = k - k = 0$

Thus $\mathbf{c}$ is orthogonal to L_k.

A Geometric Way To Solve Mathematical Programming Problems with Linear Objective Functions?

Pick any $\mathbf{x}$ in the constraint set S

Let L_k be the level curve through $\mathbf{x}$.

Then $\mathbf{c}$ is perpendicular to L_k.

Slide L_k along $\mathbf{c}$, staying parallel to L_k until we hit "last point" of

S.

Could this be the foundation for a geometric solution to a Mathematical Programming Problem with a Linear Objective Function?

Problems with the Geometric Approach

How do pick an initial point $\mathbf{x}$ in S?

How do we know when we have hit the "last" point of of S?

Important Types of Constrained Linear Objective Function Problems

1. Integer Programming

$$Z^n = \{\mathbf{x} \in R^n : \text{all components of } \mathbf{x} \text{ are integers } \}$$
$$S \subset Z^N$$

Examples: Assignment Problem, Sudoku

2. Combinatorial Programming

S is the set of all permutations of the first n positive integers

Example: Traveling Salesperson's Problem

3. Linear Programming

TSP: Traveling Salesperson's Problem

You must visit n cities, denoted 1,2,3,..., n in some order.
There is a certain cost c_{ij} in traveling from city i to city j.
Problem: Choose the order that minimizes total cost.

Order $=$ (1,2,3,4) has Cost $= c_{12} + c_{23} + c_{34}$
Order $=$ (3,1,2,4) has Cost $= c_{31} + c_{12} + c_{24}$

TSP of 50 state capitols: $n = 50$
Number of different orderings $= 50! \approx 3.04 \times 10^{64}$
Brute Force Attack? Check one billion per second: 9.6×10^{47} years.
Age of Universe: 2×10^{10} years.

Linear Programming

A is $m \times n$ matrix of constants and $\mathbf{b}$ is $n \times 1$ vector.

Constraint Set $S = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$

Example: Fromage Cheese Company Problem
$S = \{(x, y) : 30x + 12y \leq 6000, 10x + 8y \leq 2600, 4x + 8y \leq 2000, x \geq 0, y \geq 0\}$

$$A = \begin{bmatrix} 30 & 12 \\ 10 & 8 \\ 4 & 8 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 6000 \\ 2600 \\ 2000 \end{bmatrix}$$