Introduction to Operations Research

Class 3

February 17, 2023

Handouts etc.

Assignment 2 (Due Monday, February 27)
Some Problems

Your solutions to the variations of the job interview problem should include a clear statement of your suggested optimal policy.

Note on Readings for Assignment 1

Some Problems

(A) Fromage Problem

Maximize $M = f(x, y) = 4.5x + 4y$

subject to

$$30x + 12y \leq 6000$$
$$10x + 8y \leq 2600$$
$$4x + 8y \leq 2000$$
$$x \geq 0, y \geq 0$$

(B) Multivariable Calculus Problem

Maximize $M = x^2y - xy^3 - 2yz$

subject to

$$x^2 + y^2 \leq 1$$
$$2 \leq z \leq 3$$

(C) Maximize $f(x) = x^3 - 2x + \log(\sin x) - 7$ subject to $1 \leq x \leq 3$

(D) Maximize $f(x) = 1 - x^2$ subject to $-1 \leq x \leq 1$

(E) Maximize $f(x) = x^2$ subject to $x \leq 0, x \geq 1$

(F) Maximize $f(x) = x^2$ subject to $0 \leq x < 1$

(G) Maximize $f(x) = \sin x$

(H) Maximize $f(x) = 1/x$ subject to $0 < x < 1$

General Mathematical Programming Problem

Given

A set $S \subset R^n$ called the **Constraint Set**

and

A real-valued function $f : S \to R^1$ called the **Objective Function**

Want

$$\sup_{\mathbf{x} \in S} f(\mathbf{x})$$

where $M = \sup_{\mathbf{x} \in S} f(\mathbf{x})$ where "supremum of" is a number such that

(a) $f(\mathbf{x}) \leq M$ for all $\mathbf{x}$ in S and

(b) for every $\epsilon > 0$ there is some $\mathbf{x}$ in S such that $f(\mathbf{x}) > M - \epsilon$

Some Extreme Cases

If $S = \emptyset$, the problem is called **infeasible**
Example: (E) Maximize $f(x) = x^2$ subject to $x \leq 0, x \geq 1$

if $S = R^n$, the problem is called **unconstrained**
Example: (G) Maximize $f(x) = \sin x$

If $\sup_{\mathbf{x} \in S} f(\mathbf{x}) = \infty$, the problem is **unbounded**
Example: (H) Maximize $f(x) = 1/x$ subject to $0 < x < 1$

Optimal Feasible Solutions

If a vector $\mathbf{x}$ belongs to the constraint set S, then $\mathbf{x}$ is called a **feasible solution**

A vector $\mathbf{x}$ is said to **maximize** $f(\mathbf{x})$ if

$$f(\mathbf{x}) = \sup_{\mathbf{x} \in S} f(\mathbf{x})$$

and $\mathbf{x}$ is called an **optimal solution**

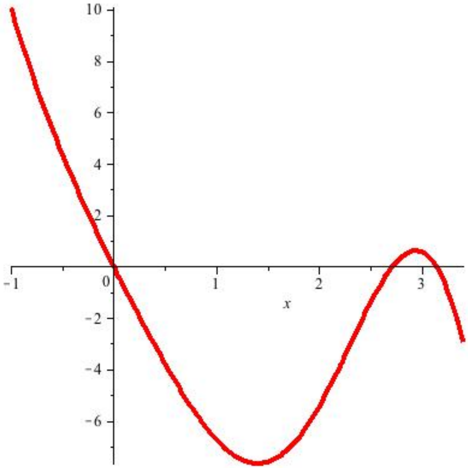
If $\mathbf{x}$ also belongs to S, then $\mathbf{x}$ is called an **optimal feasible solution**

Basic Theorem of Analysis

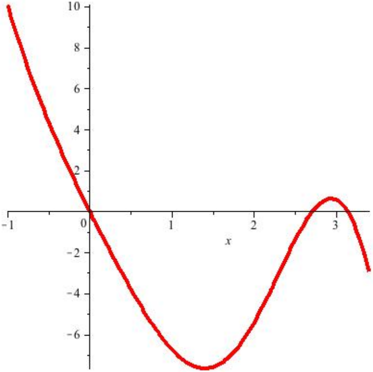
If S is a closed and bounded nonempty set in R^n and f is a continuous function, then there is an optimal feasible solution.

Note: We see this theorem first in Calculus I, then see it again in Multivariable Calculus, and finally prove it in Real Analysis (MATH 323)

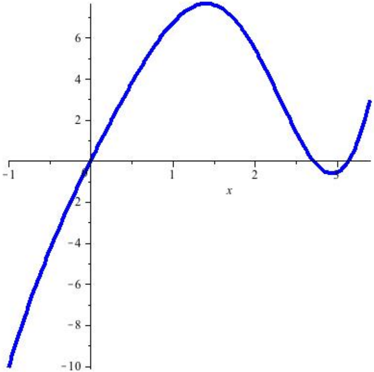
Why Don't We Talk About Minimizing Functions?



Minimum of $f(x)$ = - Maximum of $-f(x)$



Plot of f



plot of $-f$

Linear Functions

We shall begin with a focus on a particular type of objective function: Linear Functions which have the form

$$f(\mathbf{x}) = c_1x_1+c_2x_2+...+c_nx_n = (c_1, c_2, ..., c_n)^T \cdot (x_1, x_2, ..., x_n) = \mathbf{c}^T \cdot \mathbf{x}$$

To specify f, we need only know $\mathbf{c}$

Fromage: $4.5x + 4y$

$$\mathbf{c} = \begin{pmatrix} 4.5 \\ 4 \end{pmatrix}$$

Cheese Buyers Problem: $6000c + 2600s + 2000b$

$$\mathbf{c} = \begin{pmatrix} 6000 \\ 2600 \\ 2000 \end{pmatrix}$$

Transportation Problem: $464x_{11} + 513x_{12} + ... + 685x_{34}$

$$\mathbf{c} = (464, 513, ..., 685)^T$$

Basic Properties of Linear Functions

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = \mathbf{c}^T \mathbf{x}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f\mathbf{y})$$

$$f(\alpha \mathbf{x}) = \alpha f(\mathbf{x})$$

so $f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$

for any scalars α and β and any vectors $\mathbf{x}$ and $\mathbf{y}$

Proofs of Linear Function Properties

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f\mathbf{y})$$

Proof: $f(\mathbf{x}+\mathbf{y}) = \mathrm{c}^T(\mathbf{x}+\mathbf{y}) = \mathrm{c}^T\mathbf{x} + \mathrm{c}^T\mathbf{y}$ **=f(x) + f y)**

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$$

Proof: $f(\alpha\mathbf{x}) = \mathrm{c}^T(\alpha\mathbf{x}) = \alpha\mathrm{c}^T\mathbf{x} = \alpha f(\mathbf{x})$

$$f(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$$

Proof: $f(\alpha\mathbf{x}+\beta\mathbf{y}) = f(\alpha\mathbf{x}) + f(\beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$

Some Behavior of Linear Functions

If $\mathbf{c} = 0$, then $f(\mathbf{x}) = 0$ for all $\mathbf{x}$

Suppose, then, that $\mathbf{c} \neq 0$ and let $\mathbf{x} = \lambda\mathbf{c}$ where λ is a scalar.

Then $f(\mathbf{x}) = \mathbf{c}^T(\lambda\mathbf{c}) = \lambda(\mathbf{c}^T\mathbf{c}) = \lambda(c_1^2 + c_2^2 + ... + c_n^2)$ which is a **positive multiple** of λ.

By choosing λ a sufficiently large positive number, we can make $f(\mathbf{x})$ as large as we please.

By choosing λ a sufficiently large negative number, we can make $f(\mathbf{x})$ as small as we please.

THUS: **THE UNCONSTRAINED OPTIMIZATION PROBLEM FOR LINEAR OBJECTIVE FUNCTIONS IS UNBOUNDED**

THE MAXIMUM VALUE OF A LINEAR FUNCTION ON A POLYGONAL SET, IF IT EXISTS, ALWAYS OCCURS AT A VERTEX