

# Sensitivity Analysis and Dual Simplex Method

Class 19

April 3, 2023

# Dual Simplex Method

## Notes on Assignment 6

**Reminders:**  
**Team Project I Due Friday**  
**Assignment 7**  
**Due Wednesday, April 12**  
**Omit Problem 10.2-1**

# Why We Don't Use The Blackboard

Diagram illustrating the sequence of tableaux:  $\text{tableau}_1 \xrightarrow{\text{pivot}_1} \text{tableau}_2 \xrightarrow{\text{pivot}_2} \text{tableau}_3 \xrightarrow{\dots} \text{tableau}_n$ .

1) How to Set up Initial Tableau  
 2) How to pivot

1) there is a solution  
 or  
 2) no solution

$p = 10x + 6y + 2z$   
 $x \geq 0, y \geq 0, z \geq 0$   
 $2x + 2y + 3z \leq 160$   
 $5x + y + 10z \leq 100$

$p = 3x + 8y + 6z$   
 $x \geq 0, y \geq 0, z \geq 0$   
 $2x + 4y + 4z \leq 6,000$   
 $8x + 8y + 4z \leq 19,000$   
 $3x + 9y + 2z \leq 4,000$

$u, v, w$

	x	y	z	u	v	RS
u	2	2	3	1	0	160
v	5	1	10	0	1	100
p	-10	-6	-2	0	0	D.P.

	x	y	z	u	v	w	RS
u	2	4	4	1	0	0	6,000
v	8	9	4	0	1	0	19,000
w	3	4	2	0	0	1	4,000

# Systematic Sensitivity Analysis

## Some Cases to Consider

Introduction of a new variable ✓

Introduction of a new constraint ✓

Change in coefficient of nonbasic variable ✓

Change in resources (b) ✓

More Today

# Change in coefficient of a basic variable

# Relationship Between Fundamental Insight and Sensitivity

If we know the original data  $(A, \mathbf{c}, \mathbf{b})$  and which variables are in the basis, then we can determine  $B^{-1}$  and hence we can construct the entire tableau.

	Original Variables	Slack Variables	
Objective function row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$B^{-1} A$	$B^{-1}$	$B^{-1} \mathbf{b}$

Hillier and Lieberman use  $\mathbf{S}^*$  for  $B^{-1}$  if we have reached the final tableau with an optimal solution.

	Original Variables	Slack Variables	
Objective function row	$\mathbf{z}^* - \mathbf{c}$	$\mathbf{y}^*$	$Z^*$
Other rows	$\mathbf{A}^*$	$\mathbf{S}^*$	$\mathbf{b}^*$

## Final Tableau

$z - c$	$y^*$	$y_0^*$	Objective Function Row
$A^*$	$S^*$	$b^*$	Constraints

where  $S^*$  is the inverse of the matrix representing the final basis:

$$S^* = B^{-1}$$

$$b^* = B^{-1}b = S^* b$$

$$A^* = B^{-1}A = S^*A$$

$$y^* = c_B B^{-1} = c_B S^*$$

$$y_0^* = c_B B^{-1} b = y^* b$$

$$z = y^* A$$



## Changes in the Coefficient of a BASIC Variable Losing Feasibility with Changes in a Basic Variable

Original Fromage Problem

Maximize  $Z = 4.5x + 4y$  subject to

$$30x + 12y + u = 6000$$

$$10x + 8y + v = 2600$$

$$4x + 8y + w = 2000$$

$$x, y, u, v, w \geq 0$$

### Modified Problem

Maximize  $Z = 1.25x + 4y$  subject to

$$30x + 12y + u = 6000$$

$$9x + 8y + v = 2600$$

$$6x + 8y + w = 2000$$

$$x, y, u, v, w \geq 0$$

## Losing Feasibility with Changes in a Basic Variable

Maximize  $1.25x + 4y$

Subject to

$$30x + 12y \leq 6000$$

$$9x + 8y \leq 2600$$

$$6x + 8y \leq 2000$$

$$x, y, \geq 0$$

To compute how this  $x$  column would change in final tableau:  
Objective function row:

$$-\mathbf{c} + \mathbf{c}_B(x\text{-column}) = -1.25 + (0, 5/12, 1/12) \begin{pmatrix} 30 \\ 9 \\ 6 \end{pmatrix} = -\frac{5}{4} + \frac{17}{4} = 3$$

and constraint rows become:

$$B^{-1}(x - \text{column}) = \begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 30 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 9 \end{pmatrix}$$

The resulting tableau is:

	x	y	u	v	w	
	3	0	0	5/12	1/12	1250
x	1/2	0	0	1/6	-1/6	100
y	1/2	1	0	-1/12	5/24	200
u	9	0	1	-4	5/2	600

The variable  $x$  is no longer basic.

Remedy: make  $x$  basic.

Divide  $x$ -row by  $1/2$ .

Subtract 3 (new  $x$ -row) from objective function row

Subtract  $(1/2)$  (new  $x$  row) to  $y$ -row.

Subtract 9 (new  $x$ -row) to  $u$ -row

These row operations yield:

	x	y	u	v	w	
	0	0	0	$-7/12$	$13/12$	650
x	1	0	0	$1/3$	$-1/3$	200
y	0	1	0	$-1/4$	$-1/36$	100
u	0	0	1	-7	$11/2$	-1200

We have lost feasibility and optimality!

Restore Optimality first.

Let  $v$  enter the basis and  $x$  leave. Pivot on  $[1/3]$ .

The resulting tableau is

	x	y	u	v	w	
	$7/4$	0	0	0	$1/2$	1000
v	3	0	0	1	-1	600
y	$3/4$	1	0	0	$1/8$	250
u	21	0	1	0	$-3/2$	3000

We have a basic feasible optimal solution to the new problem.

**Note: Restoring optimality  
may not yield feasibility**

# Restoring Feasibility Via The Dual Simplex Method



## Restoring Feasibility Via The Dual Simplex Method

### Original Fromage Problem

$$\text{Maximize } Z = 4.5x + 4y$$

Subject to

$$30x + 12y \leq 6000$$

$$10x + 8y \leq 2600$$

$$4x + 8y \leq 2000$$

$$x, y \geq 0$$

### Original Final Tableau

	$x$	$y$	$u$	$v$	$w$	
	0	0	0	$5/12$	$1/12$	<b>1250</b>
$x$	1	0	0	$1/6$	$-1/6$	<b>100</b>
$y$	0	1	0	$-1/12$	$5/24$	<b>200</b>
$u$	0	0	1	$-4$	$5/2$	<b>600</b>

Note:  $\mathbf{c}_B = (4.5, 4, 0)$

## Original Final Tableau

	$x$	$y$	$u$	$v$	$w$	
	0	0	0	$5/12$	$1/12$	1250
$x$	1	0	0	$1/6$	$-1/6$	100
$y$	0	1	0	$-1/12$	$5/24$	200
$u$	0	0	1	$-4$	$5/2$	600

$$y^* = \left(0, \frac{5}{12}, \frac{1}{12}\right), S^* = \begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix}$$

$$\text{Change } \mathbf{b} \text{ to } \begin{pmatrix} 6000 + \Delta b_1 \\ 2600 + \Delta b_2 \\ 2000 + \Delta b_3 \end{pmatrix}$$

Then the new right hand column becomes

$$\left( \frac{\mathbf{y}^* \mathbf{b}}{S^* \mathbf{b}} \right) = \begin{pmatrix} \frac{1250 + \frac{5}{12} \Delta b_2 + \frac{1}{12} \Delta b_3}{100 + \frac{1}{6} \Delta b_2 - \frac{1}{6} \Delta b_3} \\ 200 - \frac{1}{12} \Delta b_2 + \frac{5}{24} \Delta b_3 \\ 600 + 1 \Delta b_1 - 4 \Delta b_2 + \frac{5}{2} \Delta b_3 \end{pmatrix}$$

Feasibility can be retained if

$$\begin{aligned} -600 &\leq \Delta b_1 \\ -600 &\leq \Delta b_2 \leq 150 \\ -240 &\leq \Delta b_3 \leq 600 \end{aligned}$$

$$\begin{pmatrix} 1250 + \frac{5}{12}\Delta b_2 + \frac{1}{12}\Delta b_3 \\ 100 + \frac{1}{6}\Delta b_2 - \frac{1}{6}\Delta b_3 \\ 200 - \frac{1}{12}\Delta b_2 + \frac{5}{24}\Delta b_3 \\ 600 + 1\Delta b_1 - 4\Delta b_2 + \frac{5}{2}\Delta b_3 \end{pmatrix}$$

Setting  $\Delta b_2$  and  $\Delta b_3 = 0$ :

$$600 + 1 \Delta b_1 \geq 0 \text{ implies } -600 \leq \Delta b_1$$

Setting  $\Delta b_1$  and  $\Delta b_3 = 0$ :

$$100 + \frac{1}{6}\Delta b_2 \geq 0$$

$$-600 \leq \Delta b_2$$

$$200 - \frac{1}{12}\Delta b_2 \geq 0 \text{ implies } \Delta b_2 \leq 2400 \text{ so } -600 \leq \Delta b_2 \leq 150$$

$$600 - 4\Delta b_2 \geq 0 \quad \Delta b_2 \leq 150$$

Setting  $\Delta b_1$  and  $\Delta b_2 = 0$ :

$$100 - \frac{1}{6}\Delta b_3 \geq 0$$

$$\Delta b_3 \leq 600$$

$$200 + \frac{5}{24}\Delta b_3 \geq 0 \text{ implies } -960 \leq \Delta b_3 \text{ so } -240 \leq \Delta b_3 \leq 600$$

$$600 - \frac{5}{2}\Delta b_3 \geq 0 \quad -240 \leq \Delta b_3$$

*Example 1:*  $\Delta \mathbf{b}_2 = 120$  (Swiss reserve increases by 120)

Revised Tableau becomes:

	$x$	$y$	$u$	$v$	$w$	
	0	0	0	$5/12$	$1/12$	<b>1300</b>
$x$	1	0	0	$1/6$	$-1/6$	<b>120</b>
$y$	0	1	0	$-1/12$	$5/24$	<b>190</b>
$u$	0	0	1	$-4$	$5/2$	<b>120</b>

The bfs optimal solution is

$(x, y, u, v, w) = (120, 190, 120, 0, 0)$  with  $Z = 1300$ .

Example 2:  $\Delta \mathbf{b}_2 = 180$  (Swiss reserve increases by 180)

	$x$	$y$	$u$	$v$	$w$	
	0	0	0	$5/12$	$1/12$	<b>1325</b>
$x$	1	0	0	$1/6$	$-1/6$	<b>130</b>
$y$	0	1	0	$-1/12$	$5/24$	<b>185</b>
$u$	0	0	1	$-4$	$5/2$	<b>-120</b>

This appears to give a basic, optimal solution which is **infeasible**

## Dual Simplex Method

**Idea: Use the Primal Tableau but carry out the operations as if the standard Simplex Method was being used on the Dual problem**

*To Start:* A Basic, superoptimal, infeasible solution.

*Method:* Decrease the value of the objective function, keeping the coefficients in the objective function row  $\geq 0$  until feasibility is achieved.

I. Select a basic variable with largest negative value in right hand column to **leave** the basis.

II. Select a currently nonbasic variable to **enter** the basis.

Look in the row of the exiting variable for negative coefficients.

Pick the one with the smallest ratio of the objective function row coefficient divided by absolute value of the negative coefficient.

This choice selects the one whose coefficient in the objective function row would first become negative when multiples of the exiting variable's row are subtracted from the objective function row.

Objective Function Row	...	$c$	...	$d$	...	$Z$
...						
Row of exiting variable	...	$-a$	...	$-b$	...	$-M$

Choosing  $-a$  as the pivot, we would obtain

Objective Function Row	...	0	...	$d - c(b/a)$	...	$Z$
...						
Row of exiting variable	...	1	...	$b/a$	...	$M/a$

We need  $d - c\left(\frac{b}{a}\right) \geq 0$  which is equivalent to  $\frac{d}{b} \geq \frac{c}{a}$

III. Apply Standard Iteration Row Operations to make the entering variable's column basic.



For our example:

$u$  will leave and  $v$  will enter the basis

	$x$	$y$	$u$	$v$	$w$	
	0	0	0	$5/12$	$1/12$	<b>1325</b>
$x$	1	0	0	$1/6$	$-1/6$	<b>130</b>
$y$	0	1	0	$-1/12$	$5/24$	<b>185</b>
$u$	0	0	1	$-4$	$5/2$	<b>-120</b>

Resulting Tableau

	$x$	$y$	$u$	$v$	$w$	
	0	0	$5/48$	0	$11/32$	<b>1312.5</b>
$x$	1	0	$1/24$	0	$-1/16$	<b>125</b>
$y$	0	1	$-1/48$	0	$5/32$	<b>187.5</b>
$v$	0	0	$-1/4$	1	$-5/8$	<b>30</b>

We have a basic feasible optimal solution:

$(x, y, u, v, w) = (125, 187.5, 0, 30, 0)$  with  $Z = 1312.5$ .

## Adding a New Constraint

Example: Add constraint  $2x + 3y \leq 350$  to Fromage Problem:  
Introduce slack variable  $t$  to get  $2x + 3y + t = 350$

	$x$	$y$	$u$	$v$	$w$	$t$	
	0	0	0	$5/12$	$1/12$	<b>0</b>	<b>1250</b>
$x$	1	0	0	$1/6$	$-1/6$	<b>0</b>	<b>100</b>
$y$	0	1	0	$-1/12$	$5/24$	<b>0</b>	<b>200</b>
$u$	0	0	1	$-4$	$5/2$	<b>0</b>	<b>600</b>
$t$	<b>2</b>	<b>3</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>1</b>	<b>350</b>

Make  $x$  and  $y$  basic:

Subtract 2 times  $x$ -row from  $t$ -row.

Subtract 3 times  $y$ -row from  $t$ -row.

The resulting tableau is:

	$x$	$y$	$u$	$v$	$w$	$t$	
	0	0	0	$5/12$	$1/12$	<b>0</b>	<b>1250</b>
$x$	1	0	0	$1/6$	$-1/6$	<b>0</b>	<b>100</b>
$y$	0	1	0	$-1/12$	$5/24$	<b>0</b>	<b>200</b>
$u$	0	0	1	$-4$	$5/2$	<b>0</b>	<b>600</b>
$t$	<b>0</b>	<b>0</b>	<b>0</b>	<b><math>-1/12</math></b>	<b><math>-7/24</math></b>	<b>1</b>	<b><math>-450</math></b>

Apply Dual Simplex Method to continue: (other possibilities?)

$t$  will leave

To determine entering variable:

Examine ratios:  $\frac{5/12}{1/12} = 5$  and  $\frac{1/12}{7/24} = 2/7$

The pivot element is  $-7/24$ .

The resulting tableau is:

	$x$	$y$	$u$	$v$	$w$	$t$	
	0	0	0	$11/28$	0	$2/7$	$7850/7 = 1121.4$
$x$	1	0	0	$3/14$	0	$-4/7$	$2500/7 = 357.1$
$y$	0	1	0	$-1/42$	0	$5/7$	$-850/7 = -121.4$
$u$	0	0	1	$-33/7$	0	$60/7$	$-22800/7 = -3257.1$
$w$	<b>0</b>	<b>0</b>	<b>0</b>	<b><math>3/7</math></b>	<b>1</b>	<b><math>-24/7</math></b>	<b><math>10800/7 = 1542.9</math></b>

which is basic, super optimal, infeasible.

At the next iteration,  $u$  will leave the basis and  $v$  will enter.

The pivot element is  $-33/7$ .

The resulting tableau is:

	$x$	$y$	$u$	$v$	$w$	$t$	
	0	0	$\frac{1}{12}$	0	0	1	850
$x$	1	0	$\frac{1}{22}$	0	0	$-2/11$	$2300/11 = 209.1$
$y$	0	1	$-\frac{1}{33}$	0	0	$5/11$	$-250/11 = -22.7$
$v$	0	0	$-\frac{7}{33}$	1	0	$-20/11$	$7600/11 = 690.9$
$w$	0	0	$\frac{2}{33}$	0	1	$-32/11$	$14800/11 = 1345.5$

Our current solution is still basic, superoptimal but infeasible.

At the next iteration,  $y$  will leave the basis and  $u$  will re-enter. The pivot element is  $1/33$  and the resulting tableau is

	$x$	$y$	$u$	$v$	$w$	$t$	
	0	$11/4$	0	0	0	$9/4$	$1575/2 = 787.5$
$x$	1	$3/2$	0	0	0	$1/2$	175
$u$	0	$-33$	1	0	0	$-15$	750
$v$	0	$-7$	0	1	0	$-5$	850
$w$	0	2	0	0	1	$-2$	1300

We have reached a basic, feasible, optimal solution of the modified problem:

$(x, y, u, v, w, t) = (175, 0, 750, 850, 1300, 0)$  with  $Z = 787.5$

Carlton Edward Lemke  
October 11, 1920 – April 12, 2004



1957



1978



1987

Boxer, Paratrooper, Operations Analyst, Professor  
Winner, John von Neumann Theory Prize  
The Dual Method of Solving the Linear Programming Problem,  
*Naval Research Logistics Quarterly*, Vol. 1, 1954, pp. 36-47  
Biography

THE DUAL METHOD OF SOLVING  
THE LINEAR PROGRAMMING PROBLEM \*

C. E. Lemke

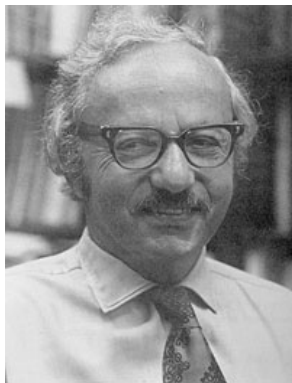
Carnegie Institute of Technology

Many problems in logistics are capable of being formulated as linear programming problems. For example, one such problem is the so-called transportation problem, which is concerned with obtaining the best way (in the sense of least time, or of least expense, etc.) of shipping stipulated quantities of materials from  $m$  origins to  $n$  destinations. This paper develops a new computational method of solving general linear programming problems.



# Next Major Topic: Network Optimization Models

# Some History of Linear Programming



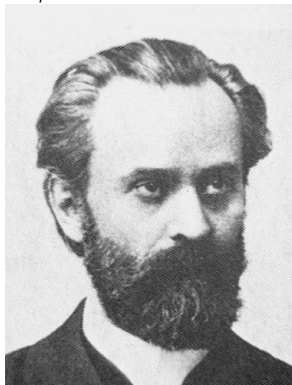
George Bernard Dantzig  
November 8, 1914 - May 13, 2005  
Father of Linear Programming





Gyula Farkas

March 28, 1847 - December 27, 1930



Farkas Lemma

There is a solution to  $Ax = b, x \geq 0$  if and only if for every  $y \geq 0$  with  $y^T b \geq 0$ , we have  $y^T A \geq 0$ .



Left to right: Tjalling C. Koopmans, George B. Dantzig, Leonid V. Kantorovich.

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1975 to Leonid Vitaliyevich Kantorovich and Tjalling C. Koopmans The photo was made in Austria, in IASA -

International Institute of System Analysis, 1970

In 1975, Tjalling C. Koopmans and Leonid V. Kantorovich received the Nobel Prize in economics for their contributions to the theory of optimum allocation of resources, that is, linear programming, or in economic terms, to its extension as linear activity analysis. Those of us familiar with the origins and development of linear programming and its impact on economic theory were amazed and disappointed that George was not included as an honoree since a Nobel Prize can be shared by up to three living recipients.

According to Michel L. Balinski, Koopmans was profoundly distressed that George had not shared in the prize. Koopmans gave a gift of \$40,000 to IIASA, the amount equal to his share of what George would have received. All three principals had worked and met at various times at IIASA. In a conversation I had with Koopmans shortly after the award, he told of his displeasure with the Nobel selection and how he had earlier written to Kantorovich suggesting that they both refuse the prize, certainly a most difficult decision for both, but especially so for Kantorovich. His work in this area received little recognition in the Soviet Union when it was first developed.

As Kantorovich noted (in a posthumous publication, 1987): "In the spring of 1939 I gave some more reports at the Polytechnic Institute and the House of Scientists, but several times met with the objection that the work used mathematical methods, and in the West the mathematical school in economics was an anti-Marxist school and mathematics in economics was a means for apologists of capitalism."

George noted in "Linear Programming and Extensions" that "Kantorovich should be credited with being the first to recognize that certain important broad classes of production problems had well-defined mathematical structures which, he believed, were amenable to practical numerical evaluation and could be numerically solved." (The book, "Activity Analysis of Production and Allocation" in the proceedings of the Cowles Commission for Research in Economics 1949 conference, edited by Koopmans was the first formal source of George's work. In 1951, Koopmans proposed the term linear programming to describe George's new mathematical model.)



Typical Problem Today  
15,000 decision variables  
3,000 constraints  
**A has 45 million entries**

# Leonid Genrikhovich Khachiyan

(May 3, 1952 - April 29, 2005) A Soviet mathematician of Armenian descent who taught Computer Science at Rutgers University. He was most famous for his Ellipsoid algorithm for linear programming, which was the first such algorithm known to have a polynomial running time. Even though this algorithm was shown to be impractical due to the high degree of the polynomial in its running time, it has inspired other randomized algorithms for convex programming and is considered a significant theoretical breakthrough.



George Dantzig and Leonid Khachyan at the Asilomar conference (1990). Dantzig invented the Simplex Method for linear optimization in 1947, and Khachyan the Ellipsoid method in 1979.

# Interior Point Method



Narendra Karmarkar  
1957 -



Karmarkar Receives Distinguished Alumnus Award from IIT, 2003

Narendra Karmarkar: In 1984, he invents a ground-breaking algorithm that improves the capabilities and lowers the cost of linear programming. The Karmarkar algorithm for linear programming made headlines not only in scientific publications but also the mass media. The method used in his algorithm has revived interest in interior methods, which are currently an important area of research in mathematical programming.

Dr. Karmarkar who has been a member of the Technical Staff, Mathematical Sciences Research Centre, AT and T Bell Labs since 1983, graduated from IIT, Mumbai, in 1978. This was followed with an M.S. from California Institute of Technology and a Ph.D. in Computer Science from the University of California, Berkeley.. Dr. Karmarkar had the distinction of receiving the Marconi International Young Scientist Award in 1985 and the Fulkerson Prize in discrete mathematics given jointly by the American Mathematical Society and Mathematical Programming Society in 1988. He received the Paris Kanellakis Award, 2000 given by The Association for Computing Machinery.