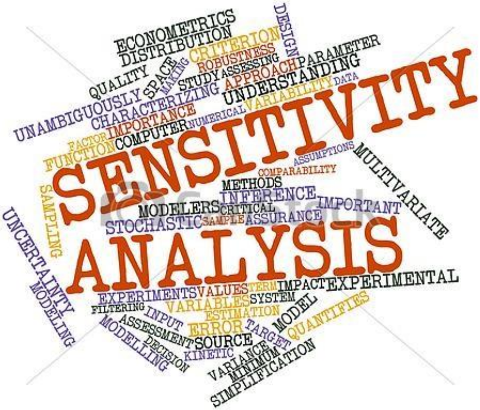


Class 18

March 31, 2023

Systematic Sensitivity Analysis Part Two

Announcements

Reminder: Team Project Progress Report

Assignment 6
Due Monday

Handout: Assignment 7
Due Wednesday April 12

Relationship Between Fundamental Insight and Sensitivity

If we know the original data (A, $\mathbf{c}$, $\mathbf{b}$) and which variables are in the basis, then we can determine B^{-1} and hence we can construct the entire tableau.

	Original Variables	Slack Variables	
Objective function row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$B^{-1} A$	B^{-1}	$B^{-1} \mathbf{b}$

Hillier and Lieberman use **S*** for B^{-1} if we have reached the final tableau with an optimal solution.

	Original Variables	Slack Variables	
Objective function row	$\mathbf{z*} - \mathbf{c}$	$\mathbf{y*}$	Z^*
Other rows	$\mathbf{A*}$	$\mathbf{S*}$	$\mathbf{b*}$

Example: Addition of a new Decision Variable

We may wish to consider attractive alternate activities.

Considering a new activity requires: introducing a new decision variable with appropriate coefficients into the objective function and new constraints into the current model.

Example: Fromage Problem
Add Aristocrat Mixture: 20 ounces of Cheddar, 12 ounces of Swiss, and 24 ounces of Brie.
Proposed selling price \$a, somewhere between \$6 and \$8.

Example: Fromage Problem

Add Aristocrat Mixture: 20 ounces of Cheddar, 12 ounces of Swiss, and 24 ounces of Brie.

Proposed selling price \$a, somewhere between \$6 and \$8.

Primal Problem becomes

$$\text{Maximize } 4.5x + 4y + az$$
$$\text{Subject to}$$
$$30x + 12y + 20z \leq 6000$$
$$10x + 8y + 12z \leq 2600$$
$$4x + 8y + 24z \leq 2000$$
$$x, y, z \geq 0$$

Switch to **Dual Problem**

$$\text{Minimize } 6000y_1 + 2600y_2 + 2000y_3$$
$$\text{Subject to}$$
$$30y_1 + 10y_2 + 4y_3 \geq 4.5$$
$$12y_1 + 8y_2 + 8y_3 \geq 4$$
$$20y_1 + 12y_2 + 24y_3 \geq a$$
$$y_1, y_2, y_3 \geq 0$$

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new decision variable
Introduction of a new constraint
Change in coefficient of a nonbasic variable
Change in resources (b)
Change in coefficient of a basic variable

General Strategy
(To deal with modifications of the original parameters)

- Calculate resulting changes in final tableau.
- Is the new solution still basic?
- Is it feasible?
- Is it optimal?
- Find new basic feasible optimal solution if necessary.
- Switching to Dual Problem may be helpful.

Final Tableau

z - c	**y***	$\mathbf{y}_0^*$	Objective Function Row
A*	**S***	**b***	Constraints

where **S*** is the inverse of the matrix representing the final basis:

$$\mathbf{S*} = B^{-1}$$
$$\mathbf{b*} = B^{-1}\mathbf{b} = \mathbf{S*}\ \mathbf{b}$$
$$\mathbf{A*} = B^{-1}A = \mathbf{S*}A$$
$$\mathbf{y*} = \mathbf{c}_B B^{-1} = \mathbf{c}_B\ \mathbf{S*}$$
$$\mathbf{y}_0\mathbf{*} = \mathbf{c}_B B^{-1}\ \mathbf{b} = \mathbf{y*b}$$
$$\mathbf{z} = \mathbf{y*}\mathrm{A}$$

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new variable ✓ Last Time
Introduction of a new constraint ✓ Last Time
Change in coefficient of a nonbasic variable
Change in resources (b)
Change in coefficient of a basic variable

Changes in the Coefficient of a Nonbasic Variables

Suppose x_j is a nonbasic variable in the optimal solution shown by the final simplex tableau.

Changes: c_j in the objective function row
a_{kj} in row k of column j

New Column j:
Objective function row: $z_j - c_j = -c_j + y * ($ column $j)$
Rest of column: **S*** (column j) $= B^{-1}$ (column j)

Changes in the Coefficient of a Nonbasic Variables

Is the New Solution Still Basic?
YES: No Basic Column Has Changed.

Is the New Solution Still Feasible?
YES: Right Hand Column Has Not Changed.

Is the New Solution Still Optimal?
YES IF coefficient of x_j in objective function row is still Non-Negative.
NO IF coefficient of x_j in objective function row is Negative.

If **NO**, the continue: Apply Simplex Method with x_j as the entering variable.

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new variable ✓
Introduction of a new constraint ✓
Change in coefficient of nonbasic variable ✓
Change in resources (b)
Change in coefficient of a basic variable

Changes in Resource Vector b

Example: Fromage with Aristrocrat at $6

Final Tableau

	x	y	**z**	u	v	w	
	0	0	**1**	0	5/12	1/12	1250
x	1	0	**-2**	0	1/6	-1/6	100
y	0	1	**4**	0	-1/12	5/24	200
u	0	0	**32**	1	-4	5/2	600

	Orignal	Change 1	Change 2
Cheddar	20	24	25
Swiss	12	10	9
Brie	24	22	22

Analysis of Change 1:

$$-6 + (0, \frac{5}{12}, \frac{1}{12}) \begin{pmatrix} 24 \\ 10 \\ 22 \end{pmatrix} = -6 + (0 + \frac{50}{12} + \frac{22}{12}) = -6 + 6 = 0$$

Continue Analysis of Change 1:

$$-6 + (0, \frac{5}{12}, \frac{1}{12}) \begin{pmatrix} 24 \\ 10 \\ 22 \end{pmatrix} = -6 + (0 + \frac{50}{12} + \frac{22}{12}) = -6 + 6 = 0$$

Multiple Optimal Solutions May Exist

Remainder of z's Column:

$$\begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 24 \\ 10 \\ 22 \end{pmatrix} = \begin{pmatrix} -2 \\ 15/4 \\ 39 \end{pmatrix}$$

The Tableau Looks Like:

	x	y	**z**	u	v	w	
	0	0	0	0	5/12	1/12	1250
x	1	0	-2	0	1/6	-1/6	100
y	0	1	15/4	0	-1/12	5/24	200
u	0	0	**[39]**	1	-4	5/2	600

	x	y	**z**	u	v	w	
	0	0	0	0	5/12	1/12	1250
x	1	0	-2	0	1/6	-1/6	100
y	0	1	15/4	0	-1/12	5/24	200
u	0	0	**[39]**	1	-4	5/2	600

z will enter the basis and u will leave.

After Iteration:

	x	y	z	u	v	w	
	0	0	0	0	5/12	1/12	1250
x	1	0	0	2/39	-1/26	-1/26	1700/13
y	0	1	0	-5/52	47/156	-5/156	1850/13
z	0	0	1	1/39	-4/39	5/78	200/13

An Alternative Optimal Solution is

$$x = \frac{1700}{13} = 130\frac{10}{13}, y = \frac{1850}{13} = 142\frac{4}{13}, z = \frac{200}{13} = 15\frac{5}{13}$$

Changes in Resource Vector b

Example: Fromage with Aristrocrat at \$6

	x	y	**z**	u	v	w	
	0	0	**1**	0	5/12	1/12	1250
x	1	0	**-2**	0	1/6	-1/6	100
y	0	1	**4**	0	-1/12	5/24	200
u	0	0	**32**	1	-4	5/2	600

	Oirignal	Change 1	**Change 2**
Cheddar	20	24	**25**
Swiss	12	10	**9**
Brie	24	22	**22**

Analysis of Change 2:

$$-6+(0,\frac{5}{12},\frac{1}{12})\begin{pmatrix}25\\9\\22\end{pmatrix}=-6+(0+\frac{45}{12}+\frac{22}{12})=-\frac{5}{12}$$

We Have Lost Optimality

Remainder of z's Column:

$$\begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 25 \\ 9 \\ 22 \end{pmatrix} = \begin{pmatrix} -13/6 \\ 23/6 \\ 44 \end{pmatrix}$$

The Tableau Looks Like:

	x	y	**z**	u	v	w	
	0	0	**-5/12**	0	5/12	1/12	1250
x	1	0	**-13/6**	0	1/6	-1/6	100
y	0	1	**23/6**	0	-1/12	5/24	200
u	0	0	[**44**]	1	-4	5/2	600

After Iteration:

	x	y	z	u	v	w	
	0	0	0	5/528	25/66	113/1056	27625/22
x	1	0	0	13/264	-1/33	-23/528	1425/11
y	0	1	0	-23/264	35/132	-5/528	1625/11
z	0	0	1	1/44	-1/11	5/88	150/11

The optimal solution is $x = 129\frac{6}{11}, y = 147\frac{8}{11}, z = 13\frac{7}{11}$ with an objective function value of about 1255.68.

Note that $x = 129, y = 147, z = 14$ actually yields a feasible integer-valued solution with objective function value $1252.50 > 1250$ of the original problem.

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new variable ✓
Introduction of a new constraint ✓
Change in coefficient of nonbasic variable ✓
Change in resources (b) ✓

Change in coefficient of a basic variable

Changes in the Coefficient of a BASIC Variable

Example: Fromage with Aristocrat at \$6

$$\text{Maximize } \mathbf{4.5}x + 4y + 6z$$
$$\text{Subject to}$$
$$30x + 12y + 20z \leq 6000 \text{ (Cheddar)}$$
$$\mathbf{10}x + 8y + 12z \leq 2600 \text{ (Swiss)}$$
$$4x + 8y + 24z \leq 2000 \text{ (Brie)}$$
$$x, y, z \geq 0$$

Final Tableau:

	x	y	z	u	v	w	
	0	0	1	0	5/12	1/12	1250
x	1	0	-2	0	1/6	-1/6	100
y	0	1	4	0	-1/12	5/24	200
u	0	0	32	1	-4	5/2	600

Suppose we decide to add 12 ounces of Swiss to the first mixture and raise the price by a dollar:

Suppose we decide to add 12 ounces of Swiss to the first mixture and raise the price by a dollar:

$$\text{Maximize } \mathbf{5.5}x + 4y + 6z$$
$$\text{Subject to}$$
$$30x + 12y + 20z \leq 6000$$
$$\mathbf{22}x + 8y + 12z \leq 2600$$
$$4x + 8y + 24z \leq 2000$$
$$x, y, z \geq 0$$

So x-column in original becomes : $\begin{pmatrix} -5.5 \\ \hline 30 \\ 22 \\ 4 \end{pmatrix}$

To compute how this x column would change in final tableau:
Objective function row:

$$-\mathbf{c} + \mathbf{c}_B(x\text{-column}) = -5.5 + (0, 5/12, 1/12)\begin{pmatrix} 30 \\ 22 \\ 4 \end{pmatrix} = 4$$

and constraint rows become:

$$B^{-1}(x - column) = \begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 30 \\ 22 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -48 \end{pmatrix}$$

The resulting tableau is:

	x	y	z	u	v	w	
	4	0	1	0	5/12	1/12	1250
x	3	0	-2	0	1/6	-1/6	100
y	-1	1	4	0	-1/12	5/24	200
u	-48	0	32	1	-4	5/2	600

The variable x is no longer basic. **Remedy: make x basic.**

Divide x-row by 3.

Subtract 4 (new x-row) from objective function row

Add 1(new x-row) to y-row.

Add 48(new x-row) to u-row

These row operations yield:

	x	y	z	u	v	w	
	0	0	11/3	0	7/36	11/36	3350/3
x	1	0	-2/3	0	1/18	-1/18	100/3
y	0	1	10/3	0	-1/36	11/72	700/3
u	0	0	0	1	-4/3	-1/6	2200

In this case, we still have feasibility and optimality.

Losing Feasibility with Changes in a Basic Variable

Original Fromage Problem

Maximize $Z = 4.5x + 4y$ subject to

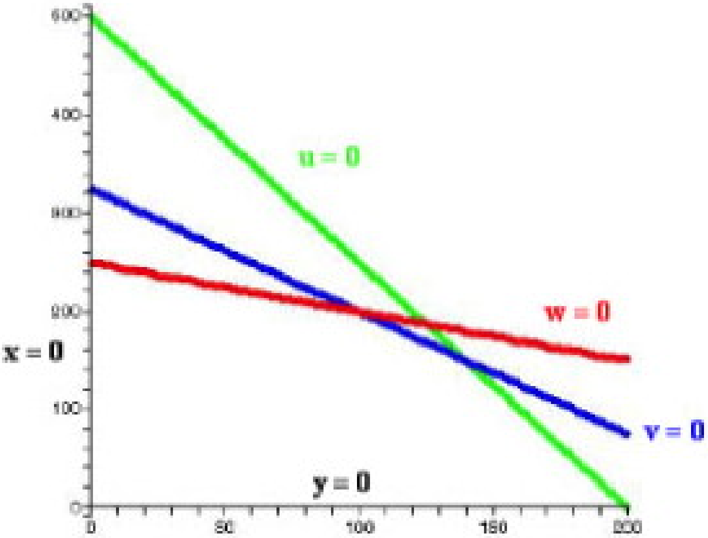
$$30x + 12y + u = 6000$$

$$10x + 8y + v = 2600$$

$$4x + 8y + w = 2000$$

$$x, y, u, v, w \geq 0$$

Basic Variables	Nonbasic	Feasible?
u,v,w	x, y	Yes
x,v,w	y, u	Yes
x,y,w	u,v	Yes
x,y,v	u,w	No
x,y,u (optimal)	v,w	Yes
y,u,v	x,w	Yes
y,u,w	x,v	No
y,v,w	x,u	No
x,u,w	y,v	No
x,u,v	y,w	No

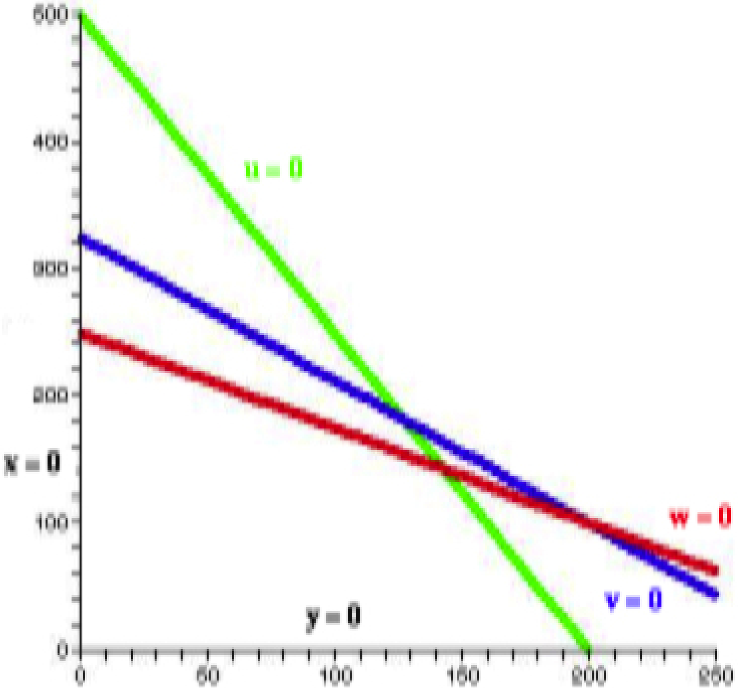

u = 0
w = 0
v = 0
x = 0
y = 0
400
200
100
0
50
100
150
200

Modified Problem

Maximize $Z = 1.25x + 4y$ subject to

$$30x + 12y + u = 6000$$
$$9x + 8y + v = 2600$$
$$6x + 8y + w = 2000$$
$$x, y, u, v, w \geq 0$$

Basic Variables	Nonbasic	Feasible?
u,v,w	x, y	Yes
x,v,w	y, u	Yes
x,y,w	u,v	No
x,y,v	u,w	No
x,y,u (original optimal)	v,w	**NO**
y,u,v	x,w	Yes
y,u,w	x,v	No
y,v,w	x,u	No
x,u,w	y,v	No
x,u,v	y,w	No



u = 0
x = 0
w = 0
v = 0
y = 0
0
100
200
300
400
500
0
50
100
150
200
250

Losing Feasibility with Changes in a Basic Variable

$$\text{Maximize } \mathbf{1.25}x + 4y$$
$$\text{Subject to}$$
$$\mathbf{30}x + 12y \leq 6000$$
$$\mathbf{9}x + 8y \leq 2600$$
$$\mathbf{6}x + 8y \leq 2000$$
$$x, y, \geq 0$$

To compute how this x column would change in final tableau:
Objective function row:

$$-\mathbf{c}+\mathbf{c}_B(x\text{-column}) = -1.25+(0, 5/12, 1/12)\begin{pmatrix} 30 \\ 9 \\ 6 \end{pmatrix} = -\frac{5}{4}+\frac{17}{4} = 3$$

and constraint rows become:

$$B^{-1}(x - column) = \begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 30 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 9 \end{pmatrix}$$

The resulting tableau is:

	x	y	u	v	w	
	3	0	0	5/12	1/12	1250
x	1/2	0	0	1/6	-1/6	100
y	1/2	1	0	-1/12	5/24	200
u	9	0	1	-4	5/2	600

The variable x is no longer basic.
Remedy: make x basic.
Divide x-row by 2.
Subtract 3 (new x-row) from objective function row
Subtract (1/2) (new x row) to y-row.
Subtract 9 (new x-row) to u-row

These row operations yield:

	x	y	u	v	w	
	0	0	0	- 7/12	13/12	650
x	1	0	0	1/3	-1/3	200
y	0	1	0	-1/4	-1/36	100
u	0	0	1	-7	11/2	-1200

We have lost feasibility and optimality!
Restore Optimality first.
Let v enter the basis and x leave. Pivot on [1/3].
The resulting tableau is

	x	y	u	v	w	
	7/4	0	0	0	1/2	1000
v	3	0	0	1	-1	600
y	3/4	1	0	0	1/8	250
u	21	0	1	0	-3/2	3000

We have a basic feasible optimal solution to the new problem.

Note: Restoring optimality may not yield feasibility

Next Time:
Restoring Feasibility Via The Dual Simplex Method

DUAL SIMPLEX METHOD

- The Dual Simplex Method, developed by C.E. Lemke, is very similar to the regular simplex method.
- The only differences lies in the criterion used for selecting a variable to enter the basis and one to leave the basis In dual simplex method, we first select the variable to leave the basis first and then the variable to enter the basis..
- In this method the solution starts from optimum but infeasible and remains infeasible until the true optimum is reached at which the solution becomes feasible.
- The advantage of this method is avoiding the artificial and surplus variables introduced in the constraints, as any constraint in the form of greater than or equal to $\geq$ is converted into la ess than or equal to $\leq$ constraint.