Systematic Sensitivity Analysis

Class 17

March 29, 2023

Introduction to LINGO

Lingo 8 Tutorial

Systematic Sensitivity Analysis

Announcements

Reminder: Project Progress Report Due Friday

Maximizing Absolute Value of a Linear Function

Problem: Maximize $|3x - 2y|$

subject to

$$30x + 12y \leq 1200$$

$$6x + 5y \leq 300$$

$$x \geq 0, y \geq 0$$

Let $W = 3x - 2y, P = |W| + W, N = |W| - W$

Then $P + N = 2|W|$ and $P - N = 2W = 6x - 4y$

New Problem: Maximize $0x + 0y + \frac{1}{2}P + \frac{1}{2}N)$

subject to

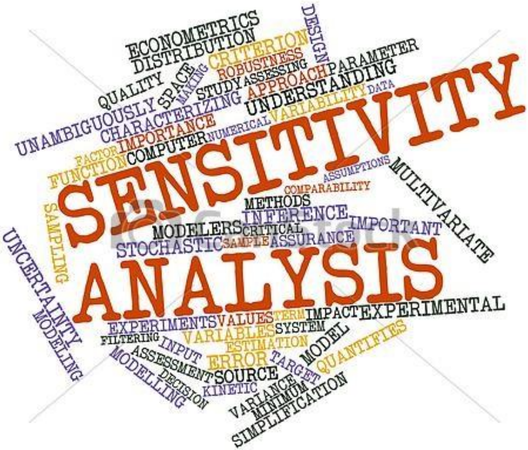
$$30x + 12y + 0P + 0N \leq 1200$$

$$6x + 5y + 0P + 0N \leq 300$$

$$6x - 4y - 1P + 1N = 0$$

$$x \geq 0, y \geq 0, P \geq 0, N \geq 0$$

Sensitivity



Relationship Between Fundamental Insight and Sensitivity

If we know the original data (A, $\mathbf{c}$, $\mathbf{b}$) and which variables are in the basis, then we can determine B^{-1} and hence we can construct the entire tableau.

	Original Variables	Slack Variables	
Objective function row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$B^{-1} A$	B^{-1}	$B^{-1} \mathbf{b}$

Hillier and Lieberman use $\mathbf{S*}$ for B^{-1} if we have reached the final tableau with an optimal solution.

	Original Variables	Slack Variables	
Objective function row	$\mathbf{z*} - \mathbf{c}$	$\mathbf{y*}$	Z^*
Other rows	$\mathbf{A*}$	$\mathbf{S*}$	$\mathbf{b*}$

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new decision variable
Introduction of a new constraint
Change in coefficient of a nonbasic variable
Change in resources (b)
Change in coefficient of a basic variable

General Strategy
(To deal with modifications of the original parameters)

- Calculate resulting changes in final tableau.
- Is the new solution still basic?
- Is it feasible?
- Is it optimal?
- Find new basic feasible optimal solution if necessary.
- Switching to Dual Problem may be helpful.

Example: Addition of a new Decision Variable

We may wish to consider attractive alternate activities.

Considering a new activity requires: introducing a new decision variable with appropriate coefficients into the objective function and new constraints into the current model.

Example: Fromage Problem
Add Aristocrat Mixture: 20 ounces of Cheddar, 12 ounces of Swiss, and 24 ounces of Brie.
Proposed selling price \$a, somewhere between \$6 and \$8.

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Primal Problem becomes

$$\text{Maximize } 4.5x + 4y + az$$

Subject to

$$30x + 12y + 20z \leq 6000$$
$$10x + 8y + 12z \leq 2600$$
$$4x + 8y + 24z \leq 2000$$
$$x, y, z \geq 0$$

Switch to **Dual Problem**

$$\text{Minimize } 6000y_1 + 2600y_2 + 2000y_3$$

Subject to

$$30y_1 + 10y_2 + 4y_3 \geq 4.5$$
$$12y_1 + 8y_2 + 8y_3 \geq 4$$
$$20y_1 + 12y_2 + 24y_3 \geq a$$
$$y_1, y_2, y_3 \geq 0$$

The Dual Problem

Minimize $6000y_1 + 2600y_2 + 2000y_3$

Subject to

$$30y_1 + 10y_2 + 4y_3 \geq 4.5$$
$$12y_1 + 8y_2 + 8y_3 \geq 4$$
$$20y_1 + 12y_2 + 24y_3 \geq a$$
$$y_1, y_2, y_3 \geq 0$$

Current Solution: $y_1 = 0, y_2 = 5/12, y_3 = 1/12$

CHECK NEW CONSTRAINT:

$$20(0) + 12(5/12) + 24(1/12) = 5 + 2 = 7 \geq a$$

Current Solution Remains Optimal for $a \leq 7$.

What happens if $a > 7$?

Alternative Approach:

Pretend the new decision variable x_j actually was in the original model with all its coefficients equal to 0 and that x_j is a nonbasic variable in the current basic feasible solution.

	x	y	z	u	v	w	
	-4.5	-4	**0**	0	0	0	0
u	30	12	**0**	1	0	0	6000
v	10	8	**0**	0	1	0	2600
w	4	8	**0**	0	0	1	2000

	x	y	z	u	v b	w	
	-4.5	-4	$-a$	0	0	0	0
u	30	12	**20**	1	0	0	6000
v	10	8	**12**	0	1	0	2600
w	4	8	**24**	0	0	1	2000

Final Tableau

	x	y	z	u	v b	w	
	0	0	**-?**	0	5/12	1/12	1250
x	1	0	**?**	0	1/6	-1/6	100
y	0	1	**?**	0	-1/12	5/24	200
u	0	0	**?**	1	-4	5/2	6000

Fundamental Insight Again

Original Tableau:

-c	**0**	0	Objective Function Row
A	I	**B**	Constraints

Choosing a Basis B, we have

$\mathbf{c}B^{-1}A$-$\mathbf{c}$	$\mathbf{c}B^{-1}$	$\mathbf{c}B^{-1}\mathbf{b}$	Objective Function Row
$B^{-1}A$	B^{-1}	$B^{-1}\mathbf{b}$	Constraints

Final Tableau

z - c	**y***	$\mathbf{y}_0^*$	Objective Function Row
A*	**S***	**b***	Constraints

Final Tableau

z - c	**y***	$\mathbf{y}_0^*$	Objective Function Row
A*	**S***	**b***	Constraints

where **S*** is the inverse of the matrix representing the final basis:

$$\mathbf{S*} = B^{-1}$$
$$\mathbf{b*} = B^{-1}\mathbf{b} = \mathbf{S*}\ \mathbf{b}$$
$$\mathbf{A*} = B^{-1}A = \mathbf{S*}A$$
$$\mathbf{y*} = \mathbf{c}_B B^{-1} = \mathbf{c}_B\ \mathbf{S*}$$
$$\mathbf{y}_0\mathbf{*} = \mathbf{c}_B B^{-1}\ \mathbf{b} = \mathbf{y*b}$$
$$\mathbf{z = y*}\mathrm{A}$$

From Fromage:
$\mathbf{y^*} = (0,\ 5/12,\ 1/12)$

$$\mathbf{S^*} = \begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix}$$

Track changes in the new z column:

$$\begin{pmatrix} 20 \\ 12 \\ 24 \end{pmatrix}$$

Objective Function Row (checking for optimality):

$$-a + (0, \frac{5}{12}, \frac{1}{12}) \begin{pmatrix} 20 \\ 12 \\ 24 \end{pmatrix} = -a + (0 + 5 + 2) = 7 - a$$

Thus: Original solution remains optimal if $a \leq 7$.

IF a $>$ 7:

We have a negative number in this column of the objective function row so this variable will enter the basis.

To find out which variable will leave the basis, we need to know the other entries in this column:

$$\begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 20 \\ 12 \\ 24 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 32 \end{pmatrix}$$

	x	y	**z**	u	v	w	
	0	0	**7-a**	0	5/12	1/12	1250
x	1	0	**-2**	0	1/6	-1/6	100
y	0	1	**4**	0	-1/12	5/24	200
u	0	0	[**32**]	1	-4	5/2	600

Ratios are $200/4 = 50$ and $600/32$ ¡ 20 so z will enter the basis and u will leave:

After Iteration:

	x	y	**z**	u	v	w	
	0	0	**0**	$\frac{a-7}{32}$	$\frac{31-3a}{24}$	$\frac{15a-89}{192}$	$\frac{4475+75a}{4}$
x	1	0	**0**	1/16	-1/12	-1/96	275/2
y	0	1	**0**	-1/8	5/12	-5/48	125
z	0	0	**1**	1/32	-1/8	5/64	75/4

Special Case:$a = 8$:

	x	y	**z**	u	v	w	
	0	0	**0**	$\frac{1}{32}$	$\frac{7}{24}$	$\frac{31}{192}$	$1268\frac{3}{4}$
x	1	0	**0**	1/16	-1/12	-1/96	275/2
y	0	1	**0**	-1/8	5/12	-5/48	125
z	0	0	**1**	1/32	-1/8	5/64	75/4

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new variable ✓
Introduction of a new constraint ✓
Change in coefficient of a nonbasic variable
Change in resources (b)
Change in coefficient of a basic variable
Introduction of a new variable

Changes in the Coefficient of a Nonbasic Variables

Suppose x_j is a nonbasic variable in the optimal solution shown by the final simplex tableau.

Changes: c_j in the objective function row
a_{kj} in row k of column j

New Column j:
Objective function row: $z_j - c_j = -c_j + y * ($ column j
Rest of column: **S*** (column j)

Changes in the Coefficient of a Nonbasic Variables

Is the New Solution Still Basic?
YES: No Basic Column Has Changed.

Is the New Solution Still Feasible?
YES: Right Hand Column Has Not Changed.

Is the New Solution Still Optimal?
YES IF coefficient of x_j in objective function row is still Non-Negative.
NO IF coefficient of x_j in objective function row is Negative.

If **NO**, the continue: Apply Simplex Method with x_j as the entering variable.

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new variable ✓
Introduction of a new constraint ✓
Change in coefficient of nonbasic variable ✓
Change in resources (b)
Change in coefficient of a basic variable

Changes in Resource Vector b

Example: Fromage with Aristrocrat at $6

Final Tableau

	x	y	**z**	u	v	w	
	0	0	**-6**	0	5/12	1/12	1250
x	1	0	**-2**	0	1/6	-1/6	100
y	0	1	**4**	0	-1/12	5/24	200
u	0	0	**32**	1	-4	5/2	600

	Oirignal	Change 1	Change 2
Cheddar	20	24	25
Swiss	12	10	9
Brie	24	22	22

Analysis of Change 1:

$$-6 + (0, \frac{5}{12}, \frac{1}{12}) \begin{pmatrix} 24 \\ 10 \\ 22 \end{pmatrix} = -6 + (0 + \frac{50}{12} + \frac{22}{12}) = -6 + 6 = 0$$

Continue Analysis of Change 1:

$$-6 + (0, \frac{5}{12}, \frac{1}{12}) \begin{pmatrix} 24 \\ 10 \\ 22 \end{pmatrix} = -6 + (0 + \frac{50}{12} + \frac{22}{12}) = -6 + 6 = 0$$

Multiple Optimal Solutions May Exist

Remainder of z's Column:

$$\begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 24 \\ 10 \\ 22 \end{pmatrix} = \begin{pmatrix} -2 \\ 15/4 \\ 39 \end{pmatrix}$$

The Tableau Looks Like:

	x	y	**z**	u	v	w	
	0	0	0	0	5/12	1/12	1250
x	1	0	-2	0	1/6	-1/6	100
y	0	1	15/4	0	-1/12	5/24	200
u	0	0	**[39]**	1	-4	5/2	600

	x	y	**z**	u	v	w	
	0	0	0	0	5/12	1/12	1250
x	1	0	-2	0	1/6	-1/6	100
y	0	1	15/4	0	-1/12	5/24	200
u	0	0	**[39]**	1	-4	5/2	600

z will enter the basis and u will leave.

After Iteration:

	x	y	z	u	v	w	
	0	0	0	0	5/12	1/12	1250
x	1	0	0	2/39	-1/26	-1/26	1700/13
y	0	1	0	-5/52	47/156	-5/156	1850/13
z	0	0	1	1/39	-4/39	5/78	200/13

An Alternative Optimal Solution is

$$x = \frac{1700}{13} = 130\frac{10}{13}, y = \frac{1850}{13} = 142\frac{4}{13}, z = \frac{200}{13} = 15\frac{5}{13}$$

Changes in Resource Vector b

Example: Fromage with Aristrocrat at \$6

	x	y	**z**	u	v	w	
	0	0	**-6**	0	5/12	1/12	1250
x	1	0	**-2**	0	1/6	-1/6	100
y	0	1	**4**	0	-1/12	5/24	200
u	0	0	**32**	1	-4	5/2	600

	Oirignal	Change 1	**Change 2**
Cheddar	20	24	**25**
Swiss	12	10	**9**
Brie	24	22	**22**

Analysis of Change 2:

$$-6 + (0, \frac{5}{12}, \frac{1}{12}) \begin{pmatrix} 25 \\ 9 \\ 22 \end{pmatrix} = -6 + (0 + \frac{45}{12} + \frac{22}{12}) = -\frac{5}{12}$$

We Have Lost Optimality

Remainder of z's Column:

$$\begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 25 \\ 9 \\ 22 \end{pmatrix} = \begin{pmatrix} -13/6 \\ 23/6 \\ 44 \end{pmatrix}$$

The Tableau Looks Like:

	x	y	**z**	u	v	w	
	0	0	**-5/12**	0	5/12	1/12	1250
x	1	0	**-13/6**	0	1/6	-1/6	100
y	0	1	**23/6**	0	-1/12	5/24	200
u	0	0	[**44**]	1	-4	5/2	600

After Iteration:

	x	y	z	u	v	w	
	0	0	0	5/528	25/66	113/1056	27625/22
x	1	0	0	13/264	-1/33	-23/528	1425/11
y	0	1	0	-23/264	35/132	-5/528	1625/11
z	0	0	1	1/44	-1/11	5/88	150/11

The optimal solution is $x = 129\frac{6}{11}, y = 147\frac{8}{11}, z = 13\frac{7}{11}$ with an objective function value of about 1255.68.

Note that $x = 129, y = 147, z = 14$ actually yields a feasible integer-valued solution with objective function value 1252.50 ¿ 1250 of the original problem.

Systematic Sensitivity Analysis

Some Cases to Consider

Introduction of a new variable ✓
Introduction of a new constraint ✓
Change in coefficient of nonbasic variable ✓
Change in resources (b) ✓

Change in coefficient of a basic variable