More on Fundamental Insight
S-O-B Method for Duals
Absolute Values in Objective Function

Class 16

March 27, 2023

Handouts

Sensible Rules for Remembering Duals
The S-O-B Method

Relationship Between Fundamental Insight and Duality

Relationship Between Fundamental Insight and Duality

Primal Problem	*Dual Problem*
Maximize $Z = c\mathbf{x}$	Minimize $W = \mathbf{y}b$
subject to	subject to
$A\mathbf{x} \leq b$	$\mathbf{y}A \geq c$
and $\mathbf{x} \geq 0$	and $\mathbf{y} \geq 0$.

Examine Objective Function Row After Any Iteration

Basic Variables	Original Variables	Slack Variables
	x_1 x_2 ... x_n	x_{n+1} x_{n+2} ...x_{n+m}
Z	z1-c1 z2 - c2 ... zn - cn	y1 y2 ... ym

where $\mathbf{y} = (y_1, y_2, ..., y_m)$ is the vector which the simplex method added to the original coefficients (0, 0, ,,,0)

$\mathbf{z} = (z_1, z_2, .., z_n)$ is the vector the simplex method added to original objective function row $(-c_1, -c_2, ..., -c_n)$

Fromage Example: Examine Objective Function Row

	Z	x	y	u	v	w	W
Original	1	-4.5	-4	0	0	0	0
Iteration 1	1	0	$-11/5$	$3/20$	0	0	900
Iteration 2	1	0	0	$-1/30$	$11/20$	0	1230
Iteration 3	1	0	0	0	$5/12$	$1/12$	1250

$$\text{Claim:} W = \mathbf{yb} = \sum_{i=1}^{m} b_i y_i$$

Iteration 1: $(3/20, 0, 0)\ \mathbf{b} = 900 + 0 + 0 = 900$
Iteration 2: $(-1/30, 11/20, 0)\ \mathbf{b} = -200 + 1430 + 0 = 1230$
Iteration 3: $(0, 5/12, 1/12)\ \mathbf{b} = 0 + 3250/3 + 500/3 = 1250$

$$\mathbf{z} = \mathbf{y}A \text{ so } z_j = \sum_{i=l}^{m} y_i a_{ij}$$

Optimality Conditions

$$z_j - c_j \geq 0 \Rightarrow z_j \geq c_j, \text{ for all } j, y_i \geq 0 \text{ for all } i$$

Hence $\mathbf{y}A \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}$

The only feasible solution of **y** problem is an optimal solution of the primal problem. Thus **cx** is the minimum possible feasible value for **yb**.

Quick Review

We can express the final Objective Function Row as $\mathbf{z}$ - $\mathbf{c}$ where $\mathbf{z}$ represents what was added:

$$\mathbf{z} - \mathbf{c} = [\ 0,\ 5/12,\ 1/12]\ A - \mathbf{c}$$

and [0, 5/12, 1/12] is the vector of shadow prices.

IDEA: We can reconstruct parts of the final tableau from other parts and from the original data.

General Case

$$\begin{array}{ccc} A, & \mathbf{x} & \mathbf{b} \\ m \text{ by } n & n \text{ by } 1 & m \text{ by } 1 \end{array}$$

Augment A with $m \times m$ identity matrix $\mathbb{I}$ and slack variables $\mathbf{x}_s$ (an $m \times 1$ vector)

$$[A, \mathbb{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b}$$

$$(m \times m + n) \; (m + n \times 1) = (m \times 1)$$

$$A \; \mathbf{x} + \mathbb{I} \; \mathbf{x}_s = b$$

(A system of m equations in $m + n$ unknowns)

The Matrix Form of Equations in Initial Tableau:

$$\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & A & \mathbb{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

After an iteration, the right hand side of the equation becomes

$$\begin{bmatrix} Z \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B B^{-1}\mathbf{b} \\ B^{-1}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

Thus original right hand side was multiplied on the left by

$$\begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix}$$

so left hand side was also multiplied by this matrix

$$\begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & A & \mathbb{I} \end{bmatrix}$$

Thus the left hand side has the form

$$\begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & A & \mathbb{I} \end{bmatrix} = \begin{bmatrix} 1 & -\mathbf{c} + \mathbf{c}_B B^{-1} A & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} A & B^{-1} \end{bmatrix}$$

and the matrix form of the equations of the tableau is

$$\begin{bmatrix} 1 & -\mathbf{c} + \mathbf{c}_B B^{-1} A & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} A & B^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B B^{-1} \mathbf{b} \\ B^{-1} \mathbf{b} \end{bmatrix}$$

Earlier:

If we know the original data (A, **c**, **b**) and which variables are in the basis, then we can determine B^{-1} and hence we can construct the entire tableau.

	Z	Original Variables	Slack Variables	
Z-row	1	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$\mathbf{0}$	$B^{-1} A$	B^{-1}	$B^{-1} \mathbf{b}$

Hillier and Lieberman use S* for B^{-1} if we have reached the final tableau with an optimal solution.

Initial Tableau for Fromage Problem

	Z	x	y	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

	Z	Original Variables	Slack Variables	
Z-row	1	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1}\mathbf{b}$
Other rows	$\mathbf{0}$	$B^{-1} A$	B^{-1}	$B^{-1}\mathbf{b}$

	Original Variables	Slack Variables		
Z-row	$\mathbf{c}_B B^{-1}A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1}\mathbf{b}$	A
Other rows	$B^{-1}A$	B^{-1}	$B^{-1}\mathbf{b}$	

Second Example

Suppose we take x, v, w as basis.

$$B = \begin{bmatrix} 30 & 0 & 0 \\ 10 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \Longrightarrow B^{-1} = \begin{bmatrix} 1/30 & 0 & 0 \\ -1/3 & 1 & 0 \\ -2/15 & 0 & 1 \end{bmatrix}$$

	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1}\mathbf{b}$
Other rows	$B^{-1} A$	B^{-1}	$B^{-1}\mathbf{b}$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 1/30 & 0 & 0 \\ -1/3 & 1 & 0 \\ -2/15 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6000 \\ 2600 \\ 2000 \end{bmatrix} = \begin{bmatrix} 200 \\ 600 \\ 1200 \end{bmatrix}$$

$$\mathbf{c}_B(B^{-1}\mathbf{b}) = \begin{bmatrix} 9/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 200 \\ 600 \\ 1200 \end{bmatrix} = 900$$

$$\mathbf{c}_B B^{-1} = \begin{bmatrix} 9/2 & 0 & 0 \end{bmatrix} B^{-1} = \begin{bmatrix} 3/20 & 0 & 0 \end{bmatrix}$$

	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1}A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1}\mathbf{b}$
Other rows	$B^{-1}A$	B^{-1}	$B^{-1}\mathbf{b}$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 200 \\ 600 \\ 1200 \end{bmatrix}, \mathbf{c}_B(B^{-1}\mathbf{b}) = 900, \mathbf{c}_B B^{-1} = \begin{bmatrix} \frac{3}{20} & 0 & 0 \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} 1/30 & 0 & 0 \\ -1/3 & 1 & 0 \\ -2/15 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & 12 \\ 10 & 8 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 \\ 0 & 4 \\ 0 & 32/5 \end{bmatrix}$$

$$\mathbf{c}_B B^{-1}A - \mathbf{c} = \begin{bmatrix} \frac{9}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 4 \\ 0 & \frac{32}{5} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & \frac{9}{5} \end{bmatrix} - \begin{bmatrix} \frac{9}{2} & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{11}{5} \end{bmatrix}$$

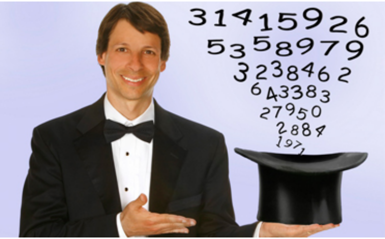
	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1}A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1}\mathbf{b}$
Other rows	$B^{-1}A$	B^{-1}	$B^{-1}\mathbf{b}$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 200 \\ 600 \\ 1200 \end{bmatrix}, \mathbf{c}_B(B^{-1}\mathbf{b}) = 900, \mathbf{c}_B B^{-1} = \begin{bmatrix} \frac{3}{20} & 0 & 0 \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} 1 & 2/5 \\ 0 & 4 \\ 0 & 32/5 \end{bmatrix}, \mathbf{c}_B B^{-1}A - \mathbf{c} = \begin{bmatrix} 0 & -11/5 \end{bmatrix}$$

$$\left[\begin{array}{cc|ccc|c} 0 & -11/5 & 3/20 & 0 & 0 & 900 \\ \hline 1 & 2/5 & 1/30 & 0 & 0 & 200 \\ 0 & 4 & -1/3 & 1 & 0 & 600 \\ 0 & 32/5 & -2/15 & 0 & 1 & 1200 \end{array}\right]$$

Sensible Rules
For Remembering Duals
The S-O-B Method
by Arthur Benjamin



Benjamin provides

- An easy way to form dual when primal has $\leq, =$ and $\geq$ constraints.
- Another way to think about the dual

S-O-B Method

S = Sensible   O = Odd   B = Bizarre

Step 1: Label Each Variable:

S if ≥ 0

O if unrestricted

B if ≤ 0

Step 2: Label Each Constraint

Maximization Problem		Minimization Problem
S if $\leq$		S if $\geq$
O if $=$		O if $=$
B if $\geq$		B if $\leq$

A CONSTRAINT in the Dual is S, O, B if the corresponding VARIABLE in the primal is S, O, B.

Example: Maximize $4x_1 + 5x_2$

subject to

$$3x_1 + 1x_2 \leq 27$$

$$5x_1 + 5x_2 = 6$$

$$6x_1 + 4x_2 \geq 6$$

$$x_1 \geq 0, x_2 \geq 0$$

Dual will have the form

Minimize $27y_1 + 6y_2 + 6y_3$

subject to

$$3y_1 + 5y_2 + 6y_3 ? 4$$

$$1y_1 + 5y_2 + 4y_3 ? 5$$

$$y_1, y_2, y_3$$

Use the S-O-B Method

Step 1: Label Variables in Primal:

Example: Maximize $4x_1 + 5x_2$

subject to

$$3x_1 + 1x_2 \leq 27$$
$$5x_1 + 5x_2 = 6$$
$$6x_1 + 4x_2 \geq 6$$
$$x_1 \geq 0, x_2 \geq 0$$

S, S

Dual will have the form

Minimize $27y_1 + 6y_2 + 6y_3$

subject to

$$3y_1 + 5y_2 + 6y_3 ? 4$$
$$1y_1 + 5y_2 + 4y_3 ? 5$$
$$y_1, y_2, y_3$$

Use the S-O-B Method

Step 2: Label Constraints in Primal:

Example: Maximize $4x_1 + 5x_2$

subject to

$3x_1 + 1x_2 \leq 27$ **S**

$5x_1 + 5x_2 = 6$ **O**

$6x_1 + 4x_2 \geq 6$ **B**

$x_1 \geq 0, x_2 \geq 0$

S,S

Dual will have the form

Minimize $27y_1 + 6y_2 + 6y_3$

subject to

$3y_1 + 5y_2 + 6y_3 ? 4$

$1y_1 + 5y_2 + 4y_3 ? 5$

y_1, y_2, y_3

Use the S-O-B Method

Label Variables and Constraints in Dual

Example: Maximize $4x_1 + 5x_2$

subject to

$3x_1 + 1x_2 \leq 27$ **S**

$5x_1 + 5x_2 = 6$ **O**

$6x_1 + 4x_2 \geq 6$ **B**

$x_1 \geq 0, x_2 \geq 0$

S,S

Dual will have the form

Minimize $27y_1 + 6y_2 + 6y_3$

subject to

$3y_1 + 5y_2 + 6y_3 ?4$ **S**

$1y_1 + 5y_2 + 4y_3 ?5$ **S**

y_1, y_2, y_3

S, O, B

Use the S-O-B Method

Put in Inequalities in Dual

Example: Maximize $4x_1 + 5x_2$

subject to

$$3x_1 + 1x_2 \leq 27 \ \textbf{S}$$
$$5x_1 + 5x_2 = 6 \ \textbf{O}$$
$$6x_1 + 4x_2 \geq 6 \ \textbf{B}$$
$$x_1 \geq 0, x_2 \geq 0$$

S,S

Dual will have the form

Minimize $27y_1 + 6y_2 + 6y_3$

subject to

$$3y_1 + 5y_2 + 6y_3 \geq 4 \ \textbf{S}$$
$$1y_1 + 5y_2 + 4y_3 \geq 5 \ \textbf{S}$$
$$y_1 \geq 0, y_2 \text{ unrestricted}, \ y_3 \leq 0$$

S, O, B

Not Quite Linear Objective Functions

Suppose we have a problem with objective function of the form

Maximize $|12x - 17y|$

Note that $|12x - 17y|$ is not a linear function.

Here's an idea: For any real number W, we have

$$P = |W| + W = \begin{cases} 2W \text{ if } x \geq 0 \\ 0 \text{ if } W < 0 \end{cases}$$

and

$$N = |W| - W = \begin{cases} 0 \text{ if } W \geq 0 \\ 2|W| \text{ if } W < 0 \end{cases}$$

Then $P + N = 2|W|$ and $P - N = 2W$

Then Maximizing $2|W|$ is equivalent to Maximizing $P + N$ and we can add a new constraint $-2W + P - N = 0$.

Next Time:

Systematic Study of Sensitivity Analysis