Another Way to Look at Duality; Parametric Programming And Prelude to Sensitivity Analysis

Class 15

Friday, March 17, 2023


HAPPY
St. Patrick's
DAY!

Notes on Exam 1

Assignment 6

Sensible Rules for Remembering Duals: The S–O–B Method

Another Way To Look At Duality

Consider Dual to Fromage Cheese Company Problem

Minimize $6000C + 2600S + 2000B$ subject to

$$30C + 10S + 4B \geq 4.5$$

$$12C + 8S + 8B \geq 4$$

$$C, S, B \geq 0$$

Let C, S, B be any feasible solution.

Multiply the first constraint by 150 and the second by 100:

$$4500C + 1500S + 600B \geq 675$$

$$1200C + 800S + 800B \geq 400.$$

Add the constraints

$$5700C + 2300S + 1400B \geq 1175.$$

But $6000C + 2600S + 2000B > 5700C + 2300S + 1400B \geq 1075$

We want to find the **best multipliers** $x, y \geq 0$ such that $x(30C + 10S + 4B) \geq 4.5x$ and $y(12C + 8S + 8B) \geq 4y$ so that $(30x + 12y)C + (10x + 8y)S + (4x + 8y)B \geq 4.5x + 4y$ is as large as possible and

$6000C + 2600S + 2000B) \geq (30x + 12y)C + (10x + 8y)S + (4x + 8y)B$

Comparing coefficients of C, S, B indicates

$6000 \geq 30x + 12y, 2600 \geq 10x + 8y, 2000 \geq 4x + 8y.$

Introduction
To
Parametric Programming
and
Sensitivity Analysis

As Usual, a Cheesy Start

Fromage Cheese Problem	*Dual Problem*
Maximize $4.5x + 4y$	Minimize $6000C + 2600S + 2000B$
subject to	subject to
$30x + 12y \leq 6000$ (Cheddar)	$30C + 10S + 4B \geq 4.5$
$10x + 8y \leq 2600$ (Swiss)	$12C + 8S + 8B \geq 4$
$4x + 8y \leq 2000$ (Brie)	
$x, y \geq 0$	$C, S, B \geq 0$
x and y are number of packages of each assortment to prepare	C, S, B are the price per ounce to offer for the cheeses

Consider the Revenue Function:

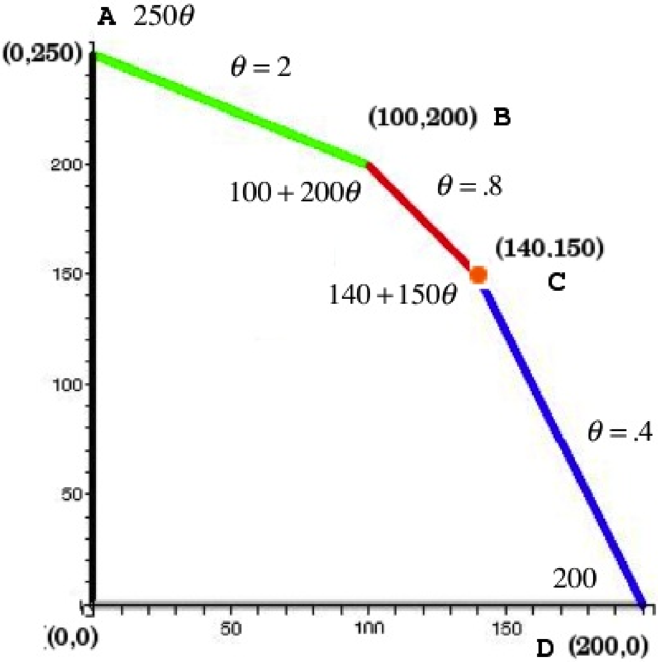
$$M = 4.5x + 4y = 4.5(x + \frac{4}{4.5}y)$$

Change in Relative Prices Charged

Revenue Function: $M = 4.5x + 4y = 4.5(x + \frac{4}{4.5}y) =$
$M = 4.5(x + \theta y)$

Vertex	Coordinates	$M = x + \theta y$	
O	(0,0)	0	
A	(0,250)	$250\ \theta$	tie at A & B when $\theta = 2$
B	(100,200)	$100 + 200\ \theta$	tie at B & C when $\theta = 4/5$
C	(140, 150)	$140 + 150\ \theta$	tie at C & D when $\theta = 2/5$
D	(200,0)	200	

(100, 200) remains optimal for $\frac{4}{5} \leq \theta \leq 2$
(100,200) maximizes $4.5x + by$ for all b such that
$\$3.60 \leq b \leq \$9.00$


A 250θ
(0,250)
$\theta = 2$
(100,200) B
200
$100 + 200\theta$
$\theta = .8$
(140,150)
150
C
$140 + 150\theta$
100
$\theta = .4$
50
200
(0,0)
50
100
150
D (200,0)

Prelude
To
Sensitivity
Analysis

Let's examine objective function row for original and final table of Fromage Problem

	Z	x	y	u	v	w	
Original	1	-4.5	-4	0	0	0	0
Final	1	0	0	0	5/12	1/12	1250

Look at

$$\left[\ \mathbf{0,\ 5/12,\ 1/12}\ \right] \begin{bmatrix} 30 & 12 & 1 & 0 & 0 & 6000 \\ 10 & 8 & 0 & 1 & 0 & 2600 \\ 4 & 8 & 0 & 0 & 1 & 2000 \end{bmatrix}$$

1 by 3 3 by 6

$$= [4.5, 4, 0, 5/12, 1/12, 1250]$$

which is exactly what got added to the original objective function row.

We can express the final Objective Function Row as $\mathbf{z}$ - $\mathbf{c}$ where $\mathbf{z}$ represents what was added:

$$\mathbf{z} - \mathbf{c} = [\,0, 5/12, 1/12]\, A - \mathbf{c}$$

and $[0, 5/12, 1/12]$ is the vector of shadow prices.

IDEA: We can reconstruct parts of the final tableau from other parts and from the original data.

Foundations of Revised Simplex Method

Fromage

$$A, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6000 \\ 2600 \\ 2000 \end{pmatrix}$$

3 by 2   2 by 1   3 by 1

Augment A with 3 x 3 identity and slack variables $\mathbf{x}_s = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$$[A, \mathbb{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b}$$

(3 x 5) (5 x 1) = (3 x 1)

$$A\ \mathbf{x} + \mathbb{I}\ \mathbf{x}_s = b$$

(A system of 3 equations in 5 unknowns)

General Case

A,	**x**	**b**
m by n	n by 1	m by 1

Augment A with $m \times m$ identity matrix $\mathbb{I}$ and slack variables $\mathbf{x}_s$ (an $m \times 1$ vector)

$$[A, \mathbb{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b}$$

$$(m \times m + n)\ (m + n \times 1) = (m \times 1)$$

$$A\ \mathbf{x} + \mathbb{I}\ \mathbf{x}_s = b$$

(A system of m equations in $m + n$ unknowns)

To find a basic solution, pick 2(n) variables to become 0 (nonbasic variables)

Then the system becomes
Fromage: 3 equations in 3 unknowns
General: m equations in m unknowns.

The coefficient matrix B of this square system of equations comes from eliminating the columns of $[A, I]$ corresponding to the nonbasic variables.
We can write the system as

$$B\mathbf{x}_B = \mathbf{b}$$

where $\mathbf{x}_B$ is the vector of basic variables obtained by eliminating the nonbasic variables from $[x, xs]^T$.

Then the solution is $\mathbf{x}_B = B^{-1}\mathbf{b}$
and the value of the objective function is
$Z = \mathbf{c}_B\mathbf{x}_B = \mathbf{c}_B(B^{-1}\mathbf{b})$

where $\mathbf{c}_B$ is the vector whose entries are the objective function coefficients for the corresponding elements of $\mathbf{x}_B$

Pay attention to dimensions:
Fromage: B^{-1} is 3 x 3, $\mathbf{b}$ is 3 x 1
General: B^{-1} is m x m, $\mathbf{b}$ is m x 1.

The Matrix Form of Equations in Initial Tableau:

$$\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & A & \mathbb{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

After an iteration, the right hand side of the equation becomes

$$\begin{bmatrix} Z \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B B^{-1}\mathbf{b} \\ B^{-1}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

Thus original right hand side was multiplied on the left by

$$\begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix}$$

so left hand side was also multiplied by this matrix

$$\begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & A & \mathbb{I} \end{bmatrix}$$

Thus the left hand side has the form

$$\begin{bmatrix} 1 & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & A & \mathbb{I} \end{bmatrix} = \begin{bmatrix} 1 & -\mathbf{c} + \mathbf{c}_B B^{-1} A & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} A & B^{-1} \end{bmatrix}$$

and the matrix form of the equations of the tableau is

$$\begin{bmatrix} 1 & -\mathbf{c} + \mathbf{c}_B B^{-1} A & \mathbf{c}_B B^{-1} \\ \mathbf{0} & B^{-1} A & B^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B B^{-1} \mathbf{b} \\ B^{-1} \mathbf{b} \end{bmatrix}$$

I Know What You Are Thinking

Thus, if we know the original data (A, $\mathbf{c}$, $\mathbf{b}$) and which variables are in the basis, then we can determine B^{-1} and hence we can construct the entire tableau.

	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$B^{-1} A$	B^{-1}	$B^{-1} \mathbf{b}$

Initial Tableau for Fromage Problem

	Z	x	y	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$B^{-1} A$	B^{-1}	$B^{-1} \mathbf{b}$

	Original Variables	Slack Variables		
Z-row	$\mathbf{c}_B B^{-1}A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1}\mathbf{b}$	An
Other rows	$B^{-1}A$	B^{-1}	$B^{-1}\mathbf{b}$	

Example

Suppose we take x, y, w as basis.

$$B = \begin{bmatrix} 30 & 12 & 0 \\ 10 & 8 & 0 \\ 4 & 8 & 1 \end{bmatrix} \Longrightarrow B^{-1} = \begin{bmatrix} 1/15 & -1/10 & 0 \\ -1/12 & 1/4 & 0 \\ 2/5 & -8/5 & 1 \end{bmatrix}$$

	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$B^{-1} A$	B^{-1}	$B^{-1} \mathbf{b}$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 1/15 & -1/10 & 0 \\ -1/12 & 1/4 & 0 \\ 2/5 & -8/5 & 1 \end{bmatrix} \begin{bmatrix} 6000 \\ 2600 \\ 2000 \end{bmatrix} = \begin{bmatrix} 140 \\ 150 \\ 240 \end{bmatrix}$$

$$\mathbf{c}_B(B^{-1}\mathbf{b}) = \begin{bmatrix} 9/2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 140 \\ 150 \\ 240 \end{bmatrix} = 630 + 600 + 0 = 1230$$

$$\mathbf{c}_B B^{-1} = \begin{bmatrix} 9/2 & 4 & 0 \end{bmatrix} B^{-1} = \begin{bmatrix} -1/30 & 11/20 & 0 \end{bmatrix}$$

	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1} A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1} \mathbf{b}$
Other rows	$B^{-1} A$	B^{-1}	$B^{-1} \mathbf{b}$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 140 \\ 150 \\ 240 \end{bmatrix}, \mathbf{c}_B(B^{-1}\mathbf{b}) = 1230, \mathbf{c}_B B^{-1} = \begin{bmatrix} -\frac{1}{30} & \frac{11}{20} & 0 \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} 1/15 & -1/10 & 0 \\ -1/12 & 1/4 & 0 \\ 2/5 & -8/5 & 1 \end{bmatrix} \begin{bmatrix} 30 & 12 \\ 10 & 8 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{c}_B B^{-1} A - \mathbf{c} = \begin{bmatrix} \frac{9}{2} & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & 4 \end{bmatrix} - \begin{bmatrix} \frac{9}{2} & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

	Original Variables	Slack Variables	
Z-row	$\mathbf{c}_B B^{-1}A - \mathbf{c}$	$\mathbf{c}_B B^{-1}$	$\mathbf{c}_B B^{-1}\mathbf{b}$
Other rows	$B^{-1}A$	B^{-1}	$B^{-1}\mathbf{b}$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 140 \\ 150 \\ 240 \end{bmatrix}, \mathbf{c}_B(B^{-1}\mathbf{b}) = 1230, \mathbf{c}_B B^{-1} = \begin{bmatrix} -\frac{1}{30} & \frac{11}{20} & 0 \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{c}_B B^{-1}A - \mathbf{c} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|ccc|c} 0 & 0 & -1/30 & 11/20 & 0 & 1230 \\ \hline 1 & 0 & 1/15 & -1/10 & 0 & 140 \\ 0 & 1 & -1/12 & 1/4 & 0 & 150 \\ 0 & 0 & 2/5 & -8/5 & 1 & 240 \end{array}\right]$$

Exercise: Do Example with x, v, w as the basis

Fromage Cheese Problem	*Dual Problem*
Maximize $4.5x + 4y$	Minimize $6000C + 2600S + 2000B$
subject to	subject to
$30x + 12y \leq 6000$ (Cheddar)	$30C + 10S + 4B \geq 4.5$
$10x + 8y \leq 2600$ (Swiss)	$12C + 8S + 8B \geq 4$
$4x + 8y \leq 2000$ (Brie)	
$x, y \geq 0$	$C, S, B \geq 0$

Dimensions: $\mathbf{x} : 2 \times 1, \mathbf{y} : 1 \times 3, A : 3 \times 2, \mathbf{c} : 1 \times 2, \mathbf{b} : 3 \times 1.$

General: $\mathbf{x} : n \times 1, \mathbf{y} : 1 \times m, A : m \times n, \mathbf{c} : 1 \times n, \mathbf{b} : m \times 1.$

Primal Problem	*Dual Problem*
Maximize $Z = c\mathbf{x}$	Minimize $W = \mathbf{y}b$
subject to	subject to
$A\mathbf{x} \leq b$	$\mathbf{y}A \geq c$
and $\mathbf{x} \geq 0$	and $\mathbf{y} \geq 0.$

Primal Problem	*Dual Problem*
Maximize $Z = cx$	Minimize $W = yb$
subject to	subject to
$Ax \leq b$	$yA \geq c$
and $x \geq 0$	and $y \geq 0$.

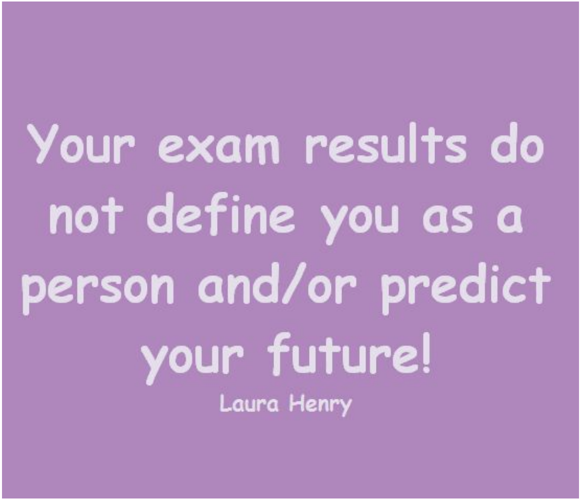
Alternative 1 for Dual	*Alternative 2 for Dual*
Minimize $W = \mathbf{b}^T\mathbf{w}$	Maximize $W = -\mathbf{b}^T\mathbf{w}$
subject to	subject to
$A^T\mathbf{w} \geq \mathbf{c}^T$	$-A^T\mathbf{w} \leq -\mathbf{c}^T$
and $\mathbf{w} \geq 0$	and $\mathbf{w} \geq 0$.
$\mathbf{w}$: column vector	$\mathbf{w}$: column vector

Announcements

Median: 89

Average: 87.6

Announcements

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