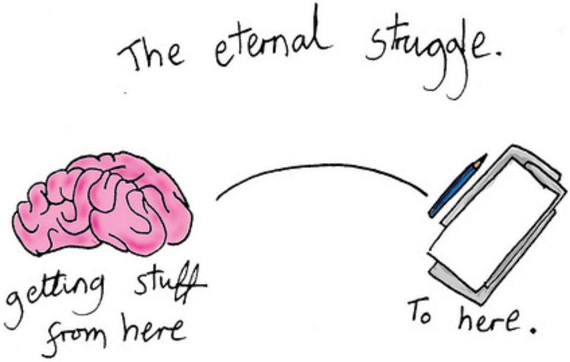
A Focus on Duality

Class 13

March 13, 2023

Introduction To Duality
Notes on Problem Set 4

Announcements



Exam 1: Wednesday at 7 PM
Warner 101

Exam 1 Details

No Time Limit

Show All Work

Double Check Your Answers

Justify Claims

Closed Book

No Calculator

Fundamental Insight: Green Numbers Record Row Operations

Initial Tableau

	Z	x	y	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

Final Tableau

	Z	x	y	u	v	w	
Z	1	0	0	**0**	**5/12**	**1/12**	1250
x	0	1	0	**0**	**1/6**	**-1/6**	100
y	0	0	1	**0**	**-1/12**	**5/24**	200
u	0	0	0	**1**	**-4**	**5/2**	600

$$(0, \frac{5}{12}, \frac{1}{12}) \cdot (6000, 2600, 2000) = \frac{0+13000+2000}{12} = \frac{15000}{12} = 1250$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{1/6} & \mathbf{-1/6} \\ \mathbf{0} & \mathbf{-1/12} & \mathbf{5/24} \\ \mathbf{1} & \mathbf{-4} & \mathbf{5/2} \end{pmatrix} \begin{pmatrix} 6000 \\ 2600 \\ 2000 \end{pmatrix} = \begin{pmatrix} 100 \\ 200 \\ 600 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{5}{12} & \frac{1}{12} \\ \mathbf{0} & \mathbf{1/6} & \mathbf{-1/6} \\ \mathbf{0} & \mathbf{-1/12} & \mathbf{5/24} \\ \mathbf{1} & \mathbf{-4} & \mathbf{5/2} \end{pmatrix} \begin{pmatrix} C \\ S \\ B \end{pmatrix} = \begin{pmatrix} \frac{5}{12}S + \frac{1}{12}B \\ \frac{S-B}{6} \\ \frac{1}{12}S + \frac{5}{24}B \\ C - 4S + \frac{5}{2}B \end{pmatrix}$$

Introduction To Duality Part II

Fromage Cheese Problem	*Dual Problem*
Maximize $4.5x + 4y$ subject to $30x + 12y \leq 6000$ (Cheddar) $10x + 8y \leq 2600$ (Swiss) $4x + 8y \leq 2000$ (Brie)	Minimize $6000C + 2600S + 2000B$ subject to $30C + 10S + 4B \geq 4.5$ $12C + 8S + 8B \geq 4$
$x, y \geq 0$	$C, S, B \geq 0$
x and y are number of packages of each assortment to prepare	C, S, B are the price per ounce to offer for the cheeses

Fromage Cheese Problem	*Dual Problem*
Maximize $4.5x + 4y$	Minimize $6000C + 2600S + 2000B$
subject to	subject to
$30x + 12y \leq 6000$ (Cheddar)	$30C + 10S + 4B \geq 4.5$
$10x + 8y \leq 2600$ (Swiss)	$12C + 8S + 8B \geq 4$
$4x + 8y \leq 2000$ (Brie)	
$x, y \geq 0$	$C, S, B \geq 0$

Dimensions: $\mathbf{x} : 2 \times 1, \mathbf{y} : 1 \times 3, A : 3 \times 2, \mathbf{c} : 1 \times 2, \mathbf{b} : 3 \times 1.$

General: $\mathbf{x} : n \times 1, \mathbf{y} : 1 \times m, A : m \times n, \mathbf{c} : 1 \times n, \mathbf{b} : m \times 1.$

Primal Problem	*Dual Problem*
Maximize $Z = c\mathbf{x}$	Minimize $W = \mathbf{y}b$
subject to	subject to
$A\mathbf{x} \leq b$	$\mathbf{y}A \geq c$
and $\mathbf{x} \geq 0$	and $\mathbf{y} \geq 0.$

Primal Problem	*Dual Problem*
Maximize $Z = cx$	Minimize $W = yb$
subject to	subject to
$Ax \leq b$	$yA \geq c$
and $x \geq 0$	and $y \geq 0$.

Note: If you like to write decision variables (unknowns) on the right, then you can write
$A^T y^T \geq c^T$ instead of $yA \geq c$

Primal Problem	*Dual Problem*
Maximize $Z = cx$ subject to $Ax \leq b$ and $x \geq 0$	Minimize $W = yb$ subject to $yA \geq c$ and $y \geq 0.$

Alternative 1 for Dual	*Alternative 2 for Dual*
Minimize $W = \mathbf{b}^T\mathbf{w}$ subject to $A^T\mathbf{w} \geq \mathbf{c}^T$ and $\mathbf{w} \geq 0$; $\mathbf{w}$: column vector	Maximize $W = -\mathbf{b}^T\mathbf{w}$ subject to $-A^T\mathbf{w} \leq -\mathbf{c}^T$ and $\mathbf{w} \geq 0.$; $\mathbf{w}$: column vector

Theorem: The dual of the dual is the primal.

Dual of the Dual is the Primal

Primal	*Alternative 2 for Dual*
Maximize $Z = \mathbf{cx}$ subject to $A\mathbf{x} \leq \mathbf{b}^T$ and $\mathbf{x} \geq 0$ $\mathbf{x}$: column vector	Maximize $W = -\mathbf{b}^T\mathbf{w}$ subject to $-A^T\mathbf{w} \leq -\mathbf{c}^T$ and $\mathbf{w} \geq 0$. $\mathbf{w}$: column vector

$$\text{Maximize } V = -(-\mathbf{c}^T)^T\mathbf{u}$$
subject to
$$-(-A^T)^T\mathbf{u} \leq -(-\mathbf{b}^T)^T$$
$$\text{and } \mathbf{u} \geq 0$$

Weak Duality Theorem

If $\mathbf{x}$ is a feasible solution to the primal problem and $\mathbf{y}$ is a feasible solution of the dual problem, then $\mathbf{cx} \leq \mathbf{yb}$.

Weak Duality Theorem

If $\mathbf{x}$ is a feasible solution to the primal problem and $\mathbf{y}$ is a feasible solution of the dual problem, then $\mathbf{cx} \leq \mathbf{yb}$.

- Corollary 1: Any feasible solution of the dual gives a bound for the primal.
- Corollary 2: Any feasible solution of the primal gives a bound for the dual.
- Corollary 3: If the primal is unbounded, then the dual is infeasible.
- Corollary 4: If primal and dual both have feasible solutions, then both have optimal solutions.
- Corollary 5: Suppose $\mathbf{x}$ is feasible for primal and $\mathbf{y}$ is feasible for dual. If $\mathbf{cx} = \mathbf{yb}$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions.

Strong Duality Property:
If $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then $\mathbf{cx}^* = \mathbf{yb}^*$.

Complementary Solutions Property

At each iteration, the simplex method simultaneously identifies a CPF solution $\mathbf{x}$ for the primal problem and a complementary solution $\mathbf{y}$ for the dual problem (in objective function row as the coefficients of the slack variables) where $\mathbf{cx} = \mathbf{yb}$. If $\mathbf{x}$ is not optimal for the primary problem, then $\mathbf{y}$ is not feasible for the dual problem.

Complementary Optimal Solutions Property: At the final iteration, the simplex method simultaneously identifies an optimal solution $\mathbf{x}^*$ for the primal problem and a complementary optimal solution $\mathbf{y}^*$ for the dual problem where $\mathbf{c}\mathbf{x}^* = \mathbf{y}^*\mathbf{b}$.

The components of $\mathbf{y}$ are the shadow prices for the primal problem.

Symmetry Property

Symmetry Property: For any primal problem and its dual problem, all relationships between them must be symmetric because the dual of this dual problem is the primal problem.

Duality Theorem

The following are the only possible relationships between the primal and dual problems:

- If one problem has feasible solutions and a bounded objective function (and therefore has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
- If one problem has feasible solutions and an unbounded objective function (and hence no optimal solution), then the other problem has no feasible solutions.
- If one problem has no feasible solutions, then the other problem has no feasible solutions or an unbounded objective function.

Fromage Cheese Company

Primal (P)	*Dual (D)*
Maximize $4.5x_1 + 4x_2$	Minimize $6000y_1 + 2600y_2 + 2000y_3$
subject to	subject to
$30x_1 + 12x_2 \leq 6000$ (Cheddar)	$30y_1 + 10y_2 + 4y_3 \geq 4.5$
$10x_1 + 8x_2 \leq 2600$ (Swiss)	$12y_1 + 8y_2 + 8y_3 \geq 4$
$4x_1 + 8x_2 \leq 2000$ (Brie)	
$x_1, x_2 \geq 0$	$y_1, y_2, y_3 \geq 0$

Initial Tableau for P

	Z	x_1	x_2	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

Constraints for Primal:

$$30(0) + 12(0) = 0 \leq 6000$$
$$10(0) + 8(0) = 0 \leq 2600$$
$$4(0) + 8(0) = 0 \leq 2000$$

Solution for Dual is $y_1 = 0, y_2 = 0, y_3 = 0$

Value of Objective function is $6000(0) + 2600(0) + 2000(0) = 0$

Nonnegativity is satisfied.

Neither constraint of dual is satisfied:

$$30(0) + 10(0) + 4(0) = 0 < 4.5 \text{ (need } \geq)$$
$$12(0) + 8(0) + 8(0) = 0 < 4 \text{ (need } \geq)$$

After First Iteration:

	Z	x_1	x_2	u	v	w	
Z	1	0	-11/5	3/20	0	0	900
x_1	0	1	2/5	1/30	0	0	200
v	0	0	4	-1/3	1	0	600
w	0	0	32/5	-2/15	0	1	1200

Constraints for Primal:

$30(200) + 12(0) = 6000 \leq 6000$ (tight)

$10(200) + 8(0) = 2000 \leq 2600$

$4(200) + 8(0) = 800 \leq 2000$

Solution for Dual is $y_1 = 3/20, y_2 = 0, y_3 = 0$

Objective function's value:$6000(3/20) + 2600(0) + 2000(0) = 900$

Nonnegativity is satisfied.

Second constraint is not satisfied:

$30(3/20) + 10(0) + 4(0) = 9/2 = 4.5$ (need $\geq$)

$12(3/20) + 8(0) + 8(0) = 9/5 = 1.8 < 4$ (need $\geq$)

After Second Iteration:

	Z	x_1	x_2	u	v	w	
Z	1	0	0	-1/30	11/20	0	1230
x_1	0	1	0	1/15	-1/10	0	140
x_2	0	0	1	-1/12	1/4	0	150
w	0	0	0	2/5	-8/3	1	240

Constraints for Primal:

$30(140) + 12(150) = 4200 + 1800 = 6000$ (tight)

$10(140) + 8(150) = 1400 + 1200 = 2600$ (tight)

$4(140) + 8(150) = 560 + 1200 = 1760 \leq 2000$

Solution for Dual is $y_1 = -1/30, y_2 = 11/20, y_3 = 0$

Value of Objective function is

6000(-1/30) + 2600(11/20) + 2000(0) = -200 + 1430 = 1230.

Nonnegativity is not satisfied.

Other constraints are satisfied:

30(-1/30) + 10(11/20) + 4(0) = 9/2= 4.5 (need $\geq$)

12((-1/30) + 8(11/20) + 8(0) = 20/5 = 4 (need $\geq$)

After Third Iteration

	Z	x_1	x_2	u	v	w	
Z	1	0	0	0	5/12	1/12	1250
x_1	0	1	0	0	1/6	-1/6	100
x_2	0	0	1	0	-1/12	5/24	200
u	0	0	0	1	-4	5/2	600

Constraints for Primal:

$30(100) + 12(200) = 3000 + 2400 = 5400 < 6000$

$10(100) + 8(200) = 1000 + 1600 = 2600$ (tight)

$4(100) + 8(200) = 400 + 1600 = 2000$ (tight)

Solution for Dual is $y_1 = 0, y_2 = 5/12, y_3 = 1/12$

Value of Objective function is $6000(0) + 2600(5/12) + 2000(1/12) = 1250$

Nonnegativity is satisfied.

Other constraints are satisfied:

$30(0) + 10(5/12) + 4(1/12) = 54/12 = 9/2 = 4.5$ (need $\geq$)

$12((0) + 8(5/12) + 8(1/12) = 48/12 = 4$ (need $\geq$)

Complementary Slackness Property

Tight or binding constraints (scarce goods) have positive shadow prices

Shadow Prices

At each iteration the value of the objective function is given by

$$Z = \mathbf{cx} = \mathbf{yb} = b_1 y_1 + b_2 y_2 + .. + b_m y_m$$

We can interpret $y_i b_i$ as the current contribution to the objective function by having b_i units of resource i available for the primal.
Thus y_i is the contribution to objective function per unit of resource i when current set of basic variables is used to obtain the primal solution.

Final Tableaux

Tableau for the Optimal Basic Feasible Solution of Primal

	Z	x	y	u	v	w	
Z	1	0	0	0	5/12	1/12	1250
x	0	1	0	0	1/6	-1/6	100
y	0	0	1	0	-1/12	5/24	200
u	0	0	0	1	-4	5/2	600

Tableau for the Optimal Basic Feasible Solution of the Dual

	Z	C	S	B	S1	S2	
Z	1	600	0	0	100	200	-1250
S	0	4	1	0	-1/6	1/12	5/12
B	0	-5/2	0	1	1/6	-5/24	1/12

Fundamental Insight: Green Numbers Record Row Operations

Initial Tableau

	Z	x	y	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

Final Tableau

	Z	x	y	u	v	w	
Z	1	0	0	**0**	**5/12**	**1/12**	1250
x	0	1	0	**0**	**1/6**	**-1/6**	100
y	0	0	1	**0**	**-1/12**	**5/24**	200
u	0	0	0	**1**	**-4**	**5/2**	600

$$(0, \frac{5}{12}, \frac{1}{12}) \cdot (6000, 2600, 2000) = \frac{0+13000+2000}{12} = \frac{15000}{12} = 1250$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{1/6} & \mathbf{-1/6} \\ \mathbf{0} & \mathbf{-1/12} & \mathbf{5/24} \\ \mathbf{1} & \mathbf{-4} & \mathbf{5/2} \end{pmatrix} \begin{pmatrix} 6000 \\ 2600 \\ 2000 \end{pmatrix} = \begin{pmatrix} 100 \\ 200 \\ 600 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{5}{12} & \frac{1}{12} \\ \mathbf{0} & \mathbf{1/6} & \mathbf{-1/6} \\ \mathbf{0} & \mathbf{-1/12} & \mathbf{5/24} \\ \mathbf{1} & \mathbf{-4} & \mathbf{5/2} \end{pmatrix} \begin{pmatrix} C \\ S \\ B \end{pmatrix} = \begin{pmatrix} \frac{5}{12}S + \frac{1}{12}B \\ \frac{S-B}{6} \\ \frac{1}{12}S + \frac{5}{24}B \\ C - 4S + \frac{5}{2}B \end{pmatrix}$$

Another Way To Look at Duality

Fromage Cheese Problem
Maximize $Z = 4.5x + 4y$
subject to
$30x + 12y \leq 6000$ (Cheddar)
$10x + 8y \leq 2600$ (Swiss)
$4x + 8y \leq 2000$ (Brie)
$x, y \geq 0$

Suppose we want to find an upper bound for Z
Multiply Cheddar constraint by $\frac{1}{6}$: $5x + 2y \leq 1000$
Multiply Brie constraint by $\frac{1}{4}$: $x + 2y \leq 500$
Now Add: $6x + 4y \leq 1500$
Then $Z = 4.5x + 4y \leq 6x + 4y \leq 1500$

Is there a **best** set of multipliers (C,S,B)?

Best Multipliers for Best Upper Bound

We want non-negative numbers C, S, B so that
$30Cx + 12Cy \leq 6000C$
$10Sx + 8Sy \leq 2600S$
$4Bx + 8By \leq 2000B$
Add:
$(30C+10S+4B)x+(12C+8S+8B)y \leq 6000C+2600S+2000B$

We need:
$4.5x \leq (30C + 10S + 4B)x$
$4y \leq (12C + 8S + 8B)y$
and
$6000C + 2600S + 2000B$ as small as possible.