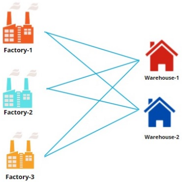
Transportation Problem III



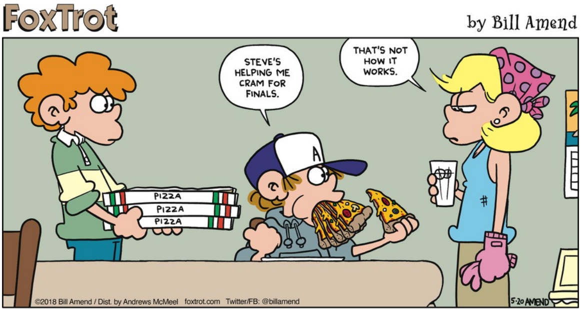
Class 34

May 10, 2023

Handouts:

Notes on Assignment 11

Final Examination
Thursday, May 18
9 am - Noon



Solving Transportation Problems

A manufacturer has $m =$ three factories at different locations F_1, F_2, F_3 that ship the product to $n =$ four warehouses in different parts of the country W_1, W_2, W_3, W_4.

She wishes to minimize the total shipping cost, while meeting the demand with the available supply.

Shipping Costs (Thousands of Dollars Per Truckload)

		Warehouses					
		W1	W2	W3	W4		
	F1	2	1	3	5	**50**	
Factories	F2	2	2	4	1	**30**	Supplies
	F3	1	4	3	2	**70**	
		40	**50**	**25**	**35**		
		Demands					

	Warehouses				
	W1	W2	W3	W4	
F1	2	1	3	5	**50**
F2	2	2	4	1	**30**
F3	1	4	3	2	**70**
	40	**50**	**25**	**35**	**150**

Let x_{ij} be the number of truckloads to send from Factory F_i to Warehouse W_j

Minimize $Z = 2x_{11} + 1x_{12} + 3x_{13} + 5x_{14} + 2x_{21} + 2x_{22} + 4x_{23} + 1x_{24} + 1x_{31} + 4x_{32} + 3x_{33} + 2x_{34}$

such that

Each factory sends out all its supplies

Each Warehouse receives its total demand.

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such that

Each factory sends out all its supplies

Each Warehouse receives its total demand.

$$x_{11} + x_{12} + x_{13} + x_{14} = 50 \qquad (A)$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 30 \qquad (B)$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 70 \qquad (C)$$
$$x_{11} + x_{21} + x_{31} = 40 \qquad (D)$$
$$x_{12} + x_{22} + x_{32} = 50 \qquad (E)$$
$$x_{13} + x_{23} + x_{33} = 25 \qquad (F)$$
$$x_{14} + x_{24} + x_{34} = 35 \qquad (G)$$
$$x_{ij} \geq 0, i = 1, 2, 3; j = 1, 2, 3, 4$$

Begin with blank tableau that has row requirements (supplies) to the right and column requirements (demands) on the bottom

				50
				30
				70
40	**50**	**25**	**35**	

Start at Northwest corner.Enter the smaller of the supply and demand. Subtract this number from supply and demand for that factory and warehouse.

Northwest Corner Rule: Step 1

$$x_{11} = 40$$

40				**10**
				30
				70
0	**50**	**25**	**35**	

The first column constraint (D) is satisfied, but the first row constraint (A) is not.

We then move to the second cell in the first row and repeat the procedure: enter the minimum of 10 and 50 and subtract 10.

Step 2:

40	10			**0**
				30
				70
0	**40**	**25**	**35**	

Now move to the second row and second column.

Enter minimum of 40 and 30

Step 3:

40	10			**0**
	30			**0**
				70
0	**10**	**25**	**35**	

What's Next? Move to the Right or Down, Depending on Which Constraint Remain to be Satisfied

Step 4:

40	10			**0**
	30			**0**
	10			**60**
0	**0**	**25**	**35**	

Step 5:

40	10			**0**
	30			**0**
	10	25		**35**
00	**00**	**00**	**35**	

Northwest Corner Rule Produced

	W1	W2	W3	W4
F1	40**B**	10**B**		
F2		30**B**		
F2		10**B**	25**B**	35**B**

We have completed the process when all constraint requirements have been met.

The process leads to at most $m+n-1$ entries: After the first choice there are $m-1$ vertical moves and $n-1$ horizontal moves for a total of $1+(m-1)+(n-1)=m+n-1$ entries corresponding to a basic feasible solution.

We need $m+n-1$ entries to have a **basic** feasible solution

Northwest Corner Rules Gives Basic Feasible Initial Solution
Without Considering Shipment Costs
Therefore, Probably Not Optimal

Vogel's Approximation Method (Discussed in H & L)
Does Pay Attention To Costs
Often Yields Optimal or Near Optimal Solutions

Our Example:
Northwest Corner Rule: 335
Vogel's Method: 205

Initial Basic Feasible Solution Via Vogel's Method

	W1	W2	W3	W4	
F1	2	1 **50**	3	5	**50**
F2	2	2	4	1 **30**	**30**
F3	1 **40**	4	3 **25**	2 **5**	**70**
	40	**50**	**25**	**35**	

Objective Function Value For This Basic Feasible Solution Is 205

Initial Basic Feasible Solution via Northwest Corner Rule

40	10			**00**
	30			**00**
	10	25	35	**00**
00	**00**	**00**	**00**	

Generating Another Basic Feasible Solution:

40	10			**50**
	30			**30**
	10	25	35	**70**
40	**50**	**25**	**35**	

Take any empty cell (one where $x_{ij} = 0$. We will make it positive and reduce some other cell to 0.

Loop Tracing:

Place a symbol y in an empty cell. Trace out a loop of alternate positive and negative *basic cells* to the empty cell. Alternate legs of the path must be perpendicular to each other and no more than two basic cells may be connected in any row or column. Consider the cell with y a positive cell.

40^-	10^+			*50*
y^+	30^-			*30*
	10	25	35	*70*
40	*50*	*25*	*35*	

The path here is $(y^+, 30^-, 10^+, 40^-)$

Enter a value for y.

Take smallest x_{ij} with a minus sign, subtract it from all the minus signed cells, and add to all the positive signed cells.

A New Basic Feasible Solution:

Subtract 30 from 40 and 30, add 30 to 10, make y 30.

40^-	10^+			*50*
y^+	30^-			*30*
	10	25	35	*70*
40	*50*	*25*	*35*	

10	40			**50**
30				**30**
	10	25	35	**70**
40	**50**	**25**	**35**	

2 40^-	1 10^+	3	3
2 y^+	2 30^-	4	1
1	4 10	3 25	2 35

Effect on Cost (per unit):

+2 -2 +1 -2 =-1 per unit shipped

Making x_{21} basic would decrease total cost by \$30.

Alternative Loop:

40^-	10^+			**50**
	30			**30**
y^+	10^-	25	35	**70**
40	**50**	**25**	**35**	

30	20			**50**
	30			**30**
10		25	35	**70**
40	**50**	**25**	**35**	

2 40^-	1 10^+	3	3
2	2 30	4	1
1 y^+	4 10^-	3 25	2 25

Effect on Cost (per unit):

+1 - 4 + 1 - 2 = -4

Making x_{31} basic would decease total cost by (-4)(10)= 40.

One More Alternative Loop:

40	10^-		y^+	**50**
	30			**30**
	10^+	25	35^-	**70**
40	**50**	**25**	**35**	

40			10	**50**
	30			**30**
	20	25	25	**70**
40	**50**	**25**	**35**	

2 40	1 10^-	3	3 y^+
2	2 30	4	1
1	4 10^+	3 25	2 35^-

Effect on Cost (per unit):

$+ 3 - 1 + 4 - 2 = 6$

Making x_{14} basic would increase total cost!

Improving A Basic Feasible Solution

Use an Iterative Procedure Like the Simplex Method to Decrease the Value of the Objective Function

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Next solution will produce

$$\bar{z} = z + y_i t_{ij}$$

where t_{ij} are **reduced costs** in the same alternating positive and negative cell path used to calculate the y.

To Decrease the Objective Function, We Only Consider t_{ij}'s Which Are Negative. We Choose the Most Negative Of These.

But How Do We Calculate t_{ij} ?

Initial Basic Feasible Solution via Northwest Corner Rule

40	10		
	30		
	10	25	35

Basic	Basic		
	Basic		
	Basic	Basic	Basic

Basic	Basic		
	Basic		
	Basic	Basic	Basic

x_{11}	x_{12}		
	x_{22}		
	x_{32}	x_{33}	x_{34}

Shipping Costs (Thousands of Dollars Per Truckload)

	W1	W2	W3	W4
F1	**2**	**1**	3	5
F2	2	**2**	4	1
F3	1	**4**	**3**	**2**

We introduce a new set of variables u_i and v_j by
$$u_i + v_j = c_{ij}$$

$$\begin{aligned}
u_1 + v_1 &= c_{11} = 2 \\
u_1 + v_2 &= c_{12} = 1 \\
u_2 + v_2 &= c_{22} = 2 \\
u_3 + v_2 &= c_{32} = 4 \\
u_3 + v_3 &= c_{33} = 3 \\
u_3 + v_4 &= c_{34} = 2
\end{aligned}$$

We have 6 linear equations in 7 unkowns

In general, there will be $m + n - 1$ equations in $m + n$ unknowns.

Set $u_1 = 0$:

$$\begin{array}{ll}
u_1 = 0 & v_1 = 2 \\
u_2 = 1 & v_2 = 1 \\
u_3 = 3 & v_3 = 0 \\
 & v_4 = -1
\end{array}$$

$u_i =$ multiple of the original row i that has been subtracted (directly or indirectly) from the original objective function row by the simplex method during all the iterations leading to the current simplex tableau.

$v_j =$ multiple of the original row $m + j$ that has been subtracted (directly or indirectly) from the original objective function row by the simplex method during all the iterations leading to the current simplex tableau.

We now compute the t_{ij} to show how to simplify their calculation. We calculate an alternative positive and negative path of costs for each nonbasic cell.

$$
\begin{aligned}
t_{13} &= c_{13} - c_{33} + c_{32} - c_{12} \\
t_{14} &= c_{14} - c_{34} + c_{32} - c_{12} \\
t_{21} &= c_{21} - c_{11} + c_{12} - c_{22} \\
t_{23} &= c_{23} - c_{33} + c_{32} - c_{22} \\
t_{24} &= c_{24} - c_{34} + c_{32} - c_{22} \\
t_{31} &= c_{31} - c_{11} + c_{12} - c_{32}
\end{aligned}
$$

The t_{ij} give the change in the objective function for one unit of x_{ij}, if $y_i = 1$ in $\bar{z} = z + y_i t_{ij}$. They play the same role as the entries of the objective function row of the simplex tableau.

Substitute $c_{ij} = u_i + v_j$ into these equations

$$t_{13} = c_{13} - c_{33} + c_{32} - c_{12}$$

$$t_{13} = c_{13} - u_3 - v_3 + u_3 + v_2 - u_1 - v_2$$

$$t_{13} = c_{13} - (u_1 + v_3)$$

similarly

$$t_{14} = c_{14} - (u_1 + v_4)$$

$$t_{21} = c_{21} - (u_2 + v_1)$$

$$t_{23} = c_{23} - (u_2 + v_3)$$

$$t_{24} = c_{24} - (u_2 + v_4)$$

$$t_{31} = c_{31} - (u_3 + v_1)$$

In general,

$$t_{ij} = c_{ij} - (u_i + v_j)$$

The current solution is

40	10			*50*
	30			*30*
	10	25	35	*70*
40	*50*	*25*	*35*	

The value of the objective function is

$z = c_{11}x_{11} + c_{12}x_{12} + c_{22}x_{22} + c_{32}x_{32} + c_{33}x_{33} + c_{34}x_{34} = 2(40) + 1(10) + 2(30) + 4(10) + 3(25) + 2(25) = 335.$

Using $t_{ij} = c_{ij} - (u_i + v_j)$

$$
\begin{aligned}
t_{13} = c_{13} - u_1 - v_3 = 3 - 0 - 0 = 3 \\
t_{14} = c_{14} - u_1 - v_4 = 5 - 0 + 1 = 6 \\
t_{21} = c_{21} - u_2 - v_1 = 2 - 1 - 2 = -1 \\
t_{23} = c_{23} - u_2 - v_3 = 4 - 1 - 0 = 3 \\
t_{24} = c_{24} - u_2 - v_4 = 1 - 1 + 1 = 1 \\
t_{31} = c_{31} - u_3 - v_1 = 1 - 3 - 2 = -4
\end{aligned}
$$

The negative t_{ij} are t_{21} and t_{31}. The most negative value is -4. We will let x_{31} enter the basis and take on a positive value by the loop method.

We take the loop path $(x_{31}^{+}, 10^{-}, 10^{+}, 40^{-})$

x_{31} Will Enter The Basis:

40^-	10^+			*50*
	30			*30*
y^+	10^-	25	35	*70*
40	*50*	*25*	*35*	

Our New Solution Is

30	20			*50*
	30			*30*
10		25	35	*70*
40	*50*	*25*	*35*	

The new value for the objective function is $z = 2(30) + 1(20) + 2(30) + 1(10) + 3(25) + 2(35) =$ 295, smaller than the preceding value of 335.

Note: $\bar{z} = z + (10)(t_{31}) = 335 + (10)(-4) = 295$

Are Any of the New t_{ij}'s Negative?

$$\begin{aligned} u_1 + v_1 &= c_{11} = 2 \\ u_1 + v_2 &= c_{12} = 1 \\ u_2 + v_2 &= c_{22} = 2 \\ u_3 + v_1 &= c_{31} = 1 \\ u_3 + v_3 &= c_{33} = 3 \\ u_3 + v_4 &= c_{34} = 2 \end{aligned} \quad \text{which leads to} \quad \begin{array}{cc} u_1 = 0 & v_1 = 2 \\ u_2 = 1 & v_2 = 1 \\ u_3 = -1 & v_3 = 4 \\ & v_4 = 3 \end{array}$$

Computing the t_{ij} for cells not in the basic solution yields

$$t_{13} = 3 - 0 - 4 = -1$$
$$t_{14} = 5 - 0 - 3 = 2$$
$$t_{21} = 2 - 1 - 2 = -1$$
$$t_{23} = 4 - 1 - 4 = -1$$
$$t_{24} = 1 - 1 - 3 = -3 \;\; t_{32} = 4 + 1 - 1 = 4$$

t_{24} is the most negative so x_{24} will enter the basis.

The loop we choose is
$(y^+, 35^-, 10^+, 30^-, 20^+, 30^-)$

30^-	20^+			*50*
	30^-		y^+	*30*
10^+		25	35^-	*70*
40	*50*	*25*	*35*	

Our New Solution is

	50			*50*
	0		30	*30*
40		25	5	*70*
40	*50*	*25*	*35*	

The new value for the objective function is $z = 295 - (30)(3) = 205$, smaller than the preceding value of 295.

Are We Done Yet?

Recompute u_i and v_j and find new values of t_ij

$$\begin{array}{l|l} u_1 = 0 & v_1 = -1 \\ u_2 = 1 & v_2 = 1 \\ u_3 = 2 & v_3 = 1 \\ & v_4 = 0 \end{array}$$

$$\begin{array}{l|l} t_{11} = 2 - 0 + 1 = 3 & t_{21} = 2 - 1 + 1 = 2 \\ t_{13} = 3 - 0 - 1 = 2 & t_{23} = 4 - 1 - 1 = 2 \\ t_{14} = 5 - 0 - 0 = 5 & t_{32} = 4 - 2 - 1 = 1 \end{array}$$

All the nonbasic t_{ij}'s are positive.

We have reached an optimal solution.

Simplex Transportation Algorithm

1. Find an initial basic feasible solution using the Northwest Corner (or Vogel) rule.
2. Calculate the u_i and v_j from the equation $u_i + v_j = c_{ij}$ for those x_{ij} in the current **basic** feasible solution.
3. Using the u_i and v_j calculated in Step 2, calculate t_{ij} for each **nonbasic** cell of the tableau.
4. If all $t_{ij} \geq 0$, the current solution is optimal.
5. If one or more $t_{ij} < 0$, then choose the most negative one and pick the corresponding x_{ij} to become positive in the new solution.
6. Calculate the new solution by find the alternating sign path using the cells in the current basic feasible solution and recalculating all x_{ij}.
7. Return To Step 2.

Demand $\neq$ Supply?

If Demand Exceeds Supply,

- Create an additional dummy row.
- Any amount shipped from this row in the optimal solution will mean that these parts of the demands cannot be satisfied.

If Supply Exceeds Demand,

- Create an additional dummy column.
- Use it to handle the amount that will not be shipped to an actual definition.
- Assume any costs c_{ij} in any dummy row or column are 0.