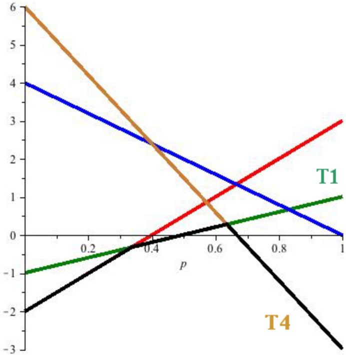
Game Theory and Duality



The Transportation Problem

Class 32

May 5, 2023

Handouts:

Notes on Assignment 11

Assignment 12

Announcements

- Final Exam
  Thursday, May 18
  9 AM - 12 Noon
- Senior Mathematics Seminar Presentations
  Next Week Watch For Announcements

Finding Your Optimal Strategy in Zero-Sum Game

- Eliminate All Dominated Strategies
- Determine "Best of the Worst"

	T_1	T_2	T_3	T_4	MIN	
S_1	5	6	20	3	**3**	
S_2	12	$\boxed{10}$	17	25	**10**	$\Leftarrow$
S_3	16	8	9	8	**8**	
S_4	13	9	6	5	**5**	
MAX	**16**	**10**	**20**	**25**		
		$\Uparrow$				

maximin $= \underline{v}$ = lower value

minimax $= \bar{v}$ = upper value

If maximin = minimax, then we have a

Saddle Point.

If there is a Saddle Point, then it is stable.
If both players know what the other will do, neither will change their strategy.

NOT EVERY GAME HAS A SADDLE POINT

A Simple Example

	T_1	T_2	Worst
S_1	8	-3	-3
S_2	-5	4	-5
Worst	8	4	

Here $\underline{v} = -3$ and $\bar{v} = 4$.
Rose can guarantee herself -3 by playing S_1.
Colin can limit her winnings to 4.
Rose tentatively selects S_1 and **Colin** selects T_2.
But if **Rose** knows **Colin** will play T_2, she should switch to S_2.
Knowing this, **Colin** switches to T_1 but.. Then **Rose** switches to S_1 causing **Colin** to...

How Should Each
Play The Game?
Choose a Strategy At Random!
But With What Probability?

Mixed Strategies

Let p be the probability **Rose** plays S_1.

The $1-p$ is the probability she plays S_2.

Similarly, q and $1-q$ would represent the probabilities that **Colin** selects T_1 and T_2, respectively.

Then the probabilities of the various outcomes are given by

	q	$1-q$
p	pq	$p(1-q)$
$1-p$	$(1-p)q$	$(1-p)(1-q)$

with payoffs

	q	$1-q$
p	8	-3
$1-p$	-5	4

Expected Value for **Rose** =

$$\mathbf{EV} = 8pq - 3p(1-q) + (-5)(1-p)q + 4(1-p)(1-q)$$

Expected Value for **Rose** $=$

$$\mathbf{EV} = 8pq - 3p(1-q) + (-5)(1-p)q + 4(1-p)(1-q)$$

$$= 20pq - 7p - 9q + 4$$

$$= (4p - \frac{9}{5})(5q - \frac{7}{4}) - \frac{63}{4} + 4$$

$$= (4p - \frac{9}{5})(5q - \frac{7}{4}) + \frac{17}{20}$$

Rose: $4p = \frac{9}{5} \Rightarrow p = \frac{9}{20}$

Colin: $5q - \frac{7}{4} \Rightarrow q = \frac{7}{20}$

	T_1	T_2	Worst
S_1	8	-3	-3
S_2	-5	4	-5
Worst	8	4	

Here $\underline{v} = -3$ and $\bar{v} = 4$.

$$EV = (4p - \frac{9}{5})(5q - \frac{7}{4}) + \frac{17}{20}$$

With $p = \frac{9}{20}$ and $q = \frac{7}{20}$,
the Expected Value of the Game is $\frac{17}{20}$

$$\mathbf{EV}_{\text{Rose}} = 20pq - 7p - 9q + 4$$

$$= (20q - 7)p - 9q + 4$$

So **Colin** should choose $q = \frac{7}{20}$

$$\mathbf{EV}_{\text{Colin}} = -(20pq - 7p - 9q + 4)$$

$$= -20pq + 7p + 9q - 4 =$$

$$(9 - 20p)q + 7p - 4$$

So **Rose** should choose $p = \frac{9}{20}$

Another Way To Determine Equilibrium Strategies

Rose's View:
Mixed Strategy vs $T_1 : 8p + (1-p)(-5) = 13p - 5$
Mixed Strategy vs $T_2 : -3p + 4(1-p) = 4 - 7p$

These payoffs are equal when

$$13p - 5 = 4 - 7p$$
$$20p = 9$$
$$p = \frac{9}{20}$$

Colin's Perspective:
Mixed Strategy vs $S_1 : 8q + (1-q)(-3) = 11q - 3$
Mixed Strategy vs $S_2 : -5q + 4(1-q) = 4 - 9q$
These payoffs are equal when

$$11q - 3 = 4 - 9q \Rightarrow 20q = 7 \Rightarrow q = \frac{7}{20}$$

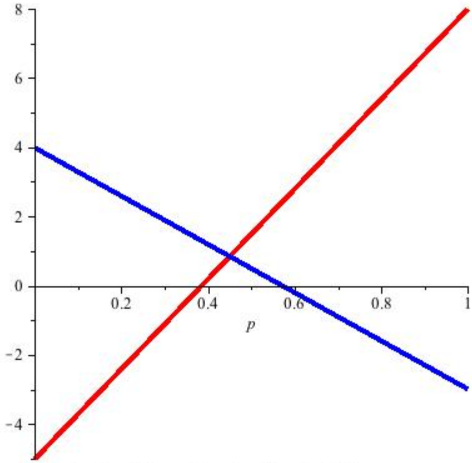
A Graphical Approach

Look at **Rose**'s Expected Payoffs:

vs $T_1 : 13p - 5$

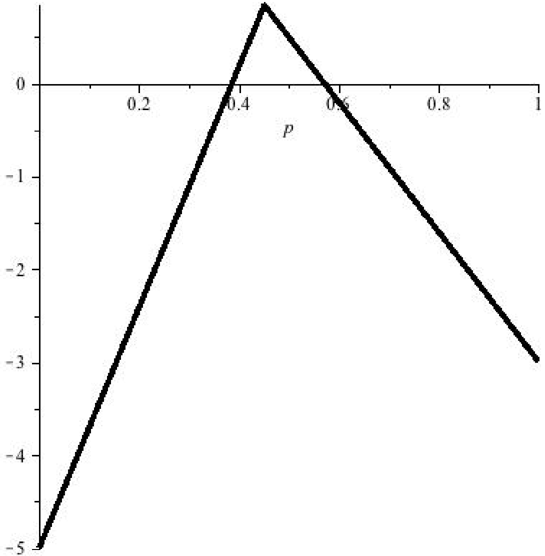
vs $T_2 : 4 - 7p$

Graph these on $0 \leq p \leq 1$.



A plot of 13 p − 5 in red and 4 − 7 p in blue

Plot $\min(13p - 5, 4 - 7p)$ for $0 \leq p \leq 1$



Another Example

	T_1	T_2	T_3	T_4
S_1	1	0	3	- 3
S_2	-1	4	-2	6

	T_1	T_2	T_3	T_4	Row Minima
S_1	1	0	3	- 3	**-3**
S_2	-1	4	-2	6	**-2**
Column Maxima	**1**	**4**	**3**	**6**	

	T_1	T_2	T_3	T_4	Row Minima
S_1	1	0	3	- 3	**-3**
S_2	-1	4	-2	6	**-2** $\Leftarrow \underline{v}$
Column Maxima	**1** $\Uparrow$ $\bar{v} = 1$	4	3	6	

	T_1	T_2	T_3	T_4	Row Minima
S_1	1	0	3	- 3	**-3**
S_2	-1	4	-2	6	**-2** $\Leftarrow \underline{v}$
Column Maxima	**1** ⇑ $\bar{v} = 1$	4	3	6	

Value of this game is somewhere between -2 and 4.

Consider Expected Payoff to **Rose** if she uses S_1 with probability p and S_2 with probability $1 - p$.

vs $T_1 : 1p + (-1)(1 - p) = 2p - 1$
vs $T_2 : 0p + 4(1 - p) = 4 - 4p$
vs $T_3 : 3p + (-2)(1 - p) = 5p - 2$
vs $T_4 : -3p + 6(1 - p) = 6 - 9p$

Expected Payoff to Rose with mixture $(p, 1-p)$:

$$\text{vs } T_1 : 2p - 1$$
$$\text{vs } T_2 : 4 - 4p$$
$$\text{vs } T_3 : 5p - 2$$
$$\text{vs } T_4 : 6 - 9p$$

Is there a single p which guarantees same expected payoff against all 4 of Colin's strategies?

T_1 and T_2: $2p - 1 = 4 - 4p \Rightarrow 6p = 5 \Rightarrow p = \frac{5}{6}$

With $p = \frac{5}{6}$:

Expected Value against $T_1 = 2(\frac{5}{6}) - 1 = 2/3$

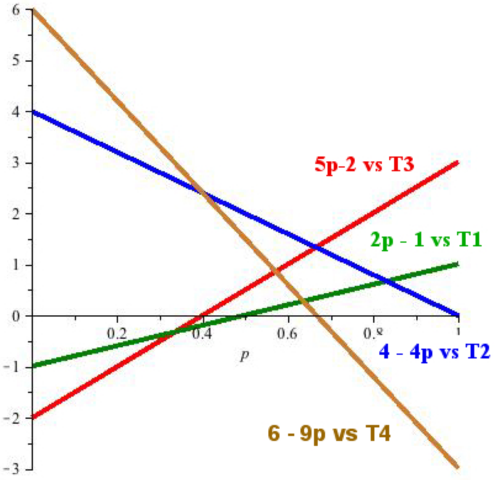
Expected Value against $T_2 = 4 - 4(\frac{5}{6}) = 2/3$

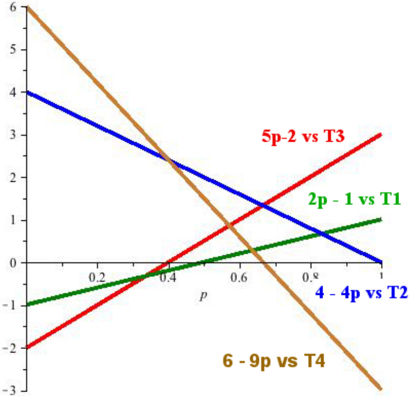
But

Expected Value against $T_3 = 5(\frac{5}{6}) - 2 = 13/6$

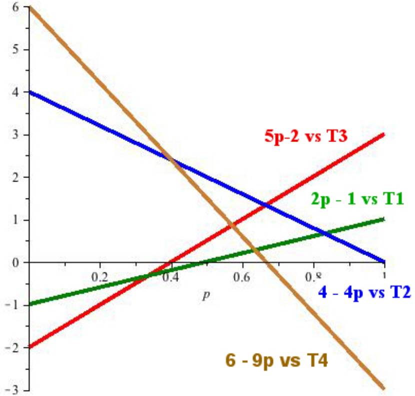
Expected Value against $T_4 = 6 - 9(\frac{5}{6}) = -3/2$

Graphical Approach



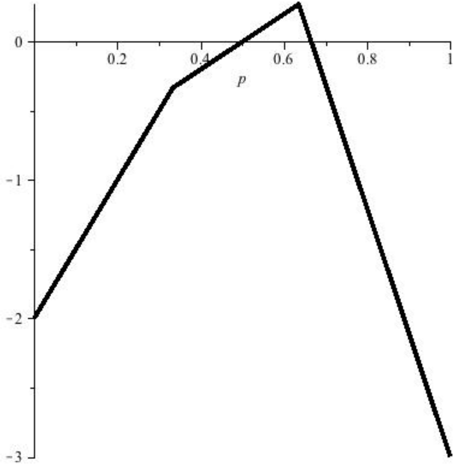


Look at bottom edge



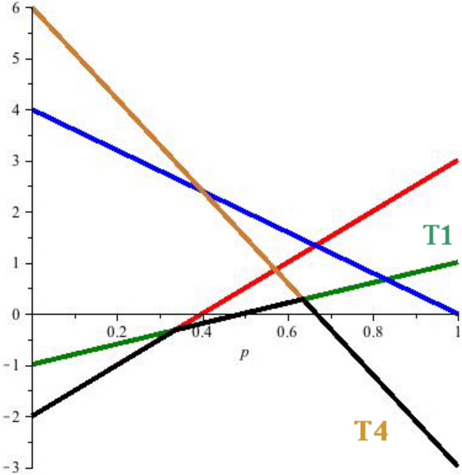
Look at bottom edge

Plot $\min(2p-1, 4-4p, 5p-2, 6-9p)$ for $0 \leq p \leq 1$



Rose can guarantee herself expected payoff of at least $\frac{3}{11}$ by choosing the strategy mixture $(\frac{7}{11}, \frac{4}{11})$.

Colin asks: Can I keep her winnings down to $\frac{3}{11}$?



6
5
4
3
2
1
0
-1
-2
-3
0.2
0.4
0.6
0.8
1
p
T1
T4

Colin wants to play a combination of T_1 and $T4$.

	q	$1-q$
	T_1	T_4
S_1	1	-3
S_2	-1	6

Expected Payoffs

vs $S_1 : 1q - 3(1-q) = 4q - 3$

vs $S_2 : -1q + 6(1-q) = 6 - 7q$

We can solve for q by

Setting $4q - 3 = \frac{3}{11}$ or

Setting $6 - 7q = \frac{3}{11}$ or

Setting $4q - 3 = 6 - 7q$

All these lead to $q = \frac{9}{11}$

Colin's Optimal Mixture is $(\frac{9}{11}, 0, 0, \frac{2}{11})$

We Can Use Graphical Approach Whenever One of the Players Has Exactly 2 Strategies

Next Time:

Connecting Game Theory

With

Linear Programming

Connecting Game Theory
With
Linear Programming

The Theory of Games and Linear Programming

Expected Payoff to Rose with mixture $(p, 1-p)$:

$$\text{vs } T_1 : 2p - 1$$
$$\text{vs } T_2 : 4 - 4p$$
$$\text{vs } T_3 : 5p - 2$$
$$\text{vs } T_4 : 6 - 9p$$

Rose can not usually hope to find a p which will make all these expected values the same

Can **Rose** pick a p so that the expected value was $\geq \frac{1}{4}$ against **each** of T_1, T_2, T_3, T_4 ?

If so, she would have an expected payoff $\geq \frac{1}{4}$ for **all** mixtures of T_1, T_2, T_3, T_4.

	T_1	T_2	T_3	T_4
S_1	1	0	3	- 3
S_2	-1	4	-2	6

Expected Payoff to **Rose** with mixture (p_1, p_2):

$$\text{vs } T_1 : 1p_1 - 1p_2$$
$$\text{vs } T_2 : 0p_1 + 4p_2$$
$$\text{vs } T_3 : 3p_1 - 2p_2$$
$$\text{vs } T_4 : -3p_1 + 6p_2$$

Rose's Question: Are there p_1 and p_2 so that all these expected payoffs are at least $\frac{1}{4}$?
Can she do even better than $\frac{1}{4}$?
What is the largest v that can be achieved?

Rose's Problem:

Maximize v such that

$$1p_1 - 1p_2 \geq v$$
$$0p_1 + 4p_2 \geq v$$
$$3p_1 - 2p_2 \geq v$$
$$-3p_1 + 6p_2 \geq v$$
$$p_1 + p_2 = 1$$
$$p_1, p_2 \geq 0$$

This is an LP Problem:

Variables: p_1, p_2, v

Surplus: s_1, s_2, s_3, s_4

Artificial: a_1, a_2, a_3, a_4, a_5

Rose's Problem	Colin's Problem
Maximize v	Minimize v
such that	such that
$1p_1 - 1p_2 \geq v$	
$0p_1 + 4p_2 \geq v$	$1q_1 + 0q_2 + 3q_3 - 3q_4 \leq v$
$3p_1 - 2p_2 \geq v$	$-1q_1 + 4q_2 - 2q_3 - +6q_4 \leq v$
$-3p_1 + 6p_2 \geq v$	$q_1 + q_2 + q_3 + q_4 = 1$
$p_1 + p_2 = 1$	
$p_1, p_2 \geq 0$	$q_1, q_2, q_3, q_4 \geq 0$
Variables: p_1, p_2, v	Variables: q_1, q_2, q_3, q_4, v
Surplus: s_1, s_2, s_3, s_4	Slacks: s_1, s_2
Artificial: a_1, a_2, a_3, a_4, a_5	Artificial: a

We don't yet know whether max/min value of v is positive or negative so these problems are not quite in standard LP form

	T_1	T_2	T_3	T_4
S_1	1	0	3	- 3
S_2	-1	4	-2	6

Largest Negative Entry is -3 So Let's Add 4 To Each Entry:

	T_1	T_2	T_3	T_4	Row Minima
S_1	5	4	7	1	**1**
S_2	3	8	2	10	**2** $\Leftarrow \underline{v} = 2$
Column Maxima	**5** $\Uparrow$ $\bar{v} = 5$	8	7	10	

$$\underline{v} = 2 \leq v \leq 5 = \bar{v}$$

	T_1	T_2	T_3	T_4
S_1	5	4	7	1
S_2	3	8	2	10

Colin's Problem: Max $-v(0q_1 + 0q_2 + 0q_3 + 0q_4 - 1v)$

such that

$$5q_1 + 4q_2 + 7q_3 + 1q_4 - v \leq 0$$
$$3q_1 + 8q_2 + 2q_3 + 10q_4 - v \leq 0$$
$$1q_1 + `q_2 + `q_3 + 1q_4 + 0v = 1$$
$$q_1, q_2, q_3, q_4, \geq 0$$
$$v \text{ unrestricted}$$

Use Benjamin's S-O-B Rule To Construct The Dual

$$\text{Max } -v = 0q_1 + 0q_2 + 0q_3 + 0q_4 - 1v$$

such that

$5q_1 + 4q_2 + 7q_3 + 1q_4 - v \leq 0$	p_1 (**S**)
$3q_1 + 8q_2 + 2q_3 + 10q_4 - v \leq 0$	p_2 (**S**)
$1q_1 + 1q_2 + 1q_3 + 1q_4 + 0v = 1$	u (**O**)
$q_1, q_2, q_3, q_4, v \geq 0$	(All **S**)
v unrestricted	**(O)**

Form Dual:

$$\text{MIN } 0p_1 + 0p_2 + 1u$$

such that

$$5p_1 + 3p_2 + 1u \geq 0$$
$$4p_1 + 8p_2 + 1u \geq 0$$
$$7p_1 + 1p_2 + 1u \geq 0$$
$$1p_1 + 10p_2 + 1u \geq 0$$
$$-p_1 - p_2 + 0u = -1$$
$$p_1 \geq 0, p_2 \geq 0, u \text{ unrestricted}$$

$$\text{MIN } 0p_1 + 0p_2 + 1u$$
such that
$$5p_1 + 3p_2 + 1u \geq 0$$
$$4p_1 + 8p_2 + 1u \geq 0$$
$$7p_1 + 1p_2 + 1u \geq 0$$
$$1p_1 + 10p_2 + 1u \geq 0$$
$$-p_1 - p_2 + 0u = -1$$
$$p_1 \geq 0, p_2 \geq 0 \text{ and } u \text{ unrestricted}$$

$$\text{MAX } v$$
such that
$$5p_1 + 3p_2 - v \geq 0$$
$$4p_1 + 8p_2 - v \geq 0$$
$$7p_1 + 1p_2 - v \geq 0$$
$$1p_1 + 10p_2 - v \geq 0$$
$$-p_1 - p_2 = 1$$
$$p_1 \geq 0, p_2 \geq 0, v = -u$$

BUT THIS IS ROSE'S PROBLEM

Consider General 2 x 4 Zero-Sum Game

	T_1	T_2	T_3	T_4
S_1	a_{11}	a_{12}	a_{13}	a_{14}
S_2	a_{21}	a_{22}	a_{23}	a_{24}

Colin's Problem: Maximize $-x_5$ such that

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 - x_5 \leq 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 - x_5 \leq 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Colin's Problem: Maximize $-x_5$ such that

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 - x_5 \leq 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 - x_5 \leq 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 x_5 \geq 0$$

Set Up Tableau

	x_1	x_2	x_3	x_4	x_5	s_1	s_2	a	
Z	0	0	0	0	1	0	0	M	0
s_1	a_{11}	a_{12}	a_{13}	a_{14}	-1	1	0	0	0
s_2	a_{21}	a_{22}	a_{23}	a_{24}	-1	0	1	0	0
a	1	1	1	1	0	0	0	1	1

Colin's Problem as LP Tableau

	x_1	x_2	x_3	x_4	x_5	s_1	s_2	a	
Z	0	0	0	0	1	0	0	M	0
s_1	a_{11}	a_{12}	a_{13}	a_{14}	-1	1	0	0	0
s_2	a_{21}	a_{22}	a_{23}	a_{24}	-1	0	1	0	0
a	1	1	1	1	0	0	0	1	1

Make a basic.

Subtract M times a row from Z row:

	x_1	x_2	x_3	x_4	x_5	s_1	s_2	a	
Z	-M	-M	-M	-M	1	0	0	0	-M
s_1	a_{11}	a_{12}	a_{13}	a_{14}	-1	1	0	0	0
s_2	a_{21}	a_{22}	a_{23}	a_{24}	-1	0	1	0	0
a	1	1	1	1	0	0	0	1	1

Simplex Algorithm: x_1 will enter the basis, driving out one of the slack variables

Observe

Colin will have 3 basic variables

One of them must be x_5, the value of the game, which must be positive.

$\Rightarrow$ There are at most 2 of the original variables that will be positive.

$\Rightarrow$ At least two of x_1, x_2, x_3, x_4 will be 0.

THE OPTIMAL STRATEGY MIXTURES IN AN m x n ZERO-SUM GAME HAVE AT MOST $\min(m, n)$ POSITIVE PROBABILITIES.

A General 3 x 4 Game

	T_1	T_2	T_3	T_4
S_1	a_{11}	a_{12}	a_{13}	a_{14}
S_2	a_{21}	a_{22}	a_{23}	a_{24}
S_3	a_{31}	a_{32}	a_{33}	a_{24}

Rose's Problem: Maximize $0x_1 + 0x_2 + 0x_3 + 1v$

subject to

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 - v \geq 0$$
$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 - v \geq 0$$
$$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 - v \geq 0$$
$$a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - v \geq 0$$
$$1x_1 + 1x_2 + 1x_3 + 0v = 1$$
$$x_1, x_2, x_3, v \geq 0$$

Rose's Problem: Maximize $0x_1 + 0x_2 + 0x_3 + 1v$

subject to

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 - v \geq 0$$
$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 - v \geq 0$$
$$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 - v \geq 0$$
$$a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - v \geq 0$$
$$1x_1 + 1x_2 + 1x_3 + 0v = 1$$
$$x_1, x_2, x_3, v \geq 0$$

Colin's Problem: Minimize $0y_1 + 0y_2 + 0y_3 + 0y_4 + 1u$

subject to

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + a_{14}y_4 - u \leq 0$$
$$a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + a_{24}y_4 - u \leq 0$$
$$a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + a_{34}y_4 - u \leq 0$$
$$1y_1 + 1y_2 + 1y_3 + 1y_4 + 0u = 1$$
$$y_1, y_2, y_3, y_4, u \geq 0$$

Rose's Problem: Minimize $0x_1 + 0x_2 + 0x_3 + 1u$
subject to

$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + u \geq 0$ **S**

$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + u \geq 0$ **S**

$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + u \geq 0$ **S**

$a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + u \geq 0$ **S**

$1x_1 + 1x_2 + 1x_3 + 0u = 1$ **O**

$x_1, x_2, x_3 \geq 0$

Colin's Objective Function:

Minimize $0y_1 + 0y_2 + 0y_3 + 0y_4 + 1u$

Rewrite as

Maximize $0y_1 + 0y_2 + 0y_3 + 0y_4$- **1u**

Then let $v = -u$

Maximize $0y_1 + 0y_2 + 0y_3 + 0y_4$**+1v**

$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + a_{14}y_4 + v \leq 0$

$a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + a_{24}y_4 - +v \leq 0$

$a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + a_{34}y_4 - +v \leq 0$

$1y_1 + 1y_2 + 1y_3 + 1y_4 + 0u = 1$

$y_1, y_2, y_3, y_4 \geq 0$, No restriction on v

Colin's Problem : Maximize $0y_1 + 0y_2 + 0y_3 + 0y_4 \mathbf{+1v}$

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + a_{14}y_4 + v \leq 0 \ \mathbf{S}$$
$$a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + a_{24}y_4 - +v \leq 0 \ \mathbf{S}$$
$$a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + a_{34}y_4 - +v \leq 0 \ \mathbf{S}$$
$$1y_1 + 1y_2 + 1y_3 + 1y_4 + 0u = 1 \ \mathbf{O}$$

$y_1, y_2, y_3, y_4 \geq 0$, No restriction on v

S, S, S, S, O

Use S-O-B Rule to Form Dual

Rose's Problem: Minimize $0x_1 + 0x_2 + 0x_3 + 1u$

subject to

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + u \geq 0 \ \mathbf{S}$$
$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + u \geq 0 \ \mathbf{S}$$
$$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + u \geq 0 \ \mathbf{S}$$
$$a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + u \geq 0 \ \mathbf{S}$$
$$1x_1 + 1x_2 + 1x_3 + 0u = 1 \ \mathbf{O}$$
$$x_1, x_2, x_3 \geq 0$$

f **You can solve m x n zero-sum games with the simplex algorithm!**

When both *m* and *n* are larger than 2, you may wish to use IOR Tutorial.

Think carefully about whether you want to solve Rose's problem or Colin's problem.

Game 1

	T_1	T_2
S_1	9	-3
S_2	-5	4

Game 2

	T_1	T_2	T_3	T_4
S_1	1	0	3	- 3
S_2	-1	4	-2	6