Introduction to Game Theory III

Class 31

May 3, 2023

Finding Your Optimal Strategy in Zero-Sum Game

- Eliminate All Dominated Strategies
- Determine "Best of the Worst"

	T_1	T_2	T_3	T_4	MIN	
S_1	5	6	20	3	**3**	
S_2	12	$\boxed{10}$	17	25	**10**	$\Leftarrow$
S_3	16	8	9	8	**8**	
S_4	13	9	6	5	**5**	
MAX	**16**	**10**	**20**	**25**		
		$\Uparrow$				

$$\text{maximin} = \underline{v} = \text{lower value}$$

$$\text{minimax} = \bar{v} = \text{upper value}$$

If maximin = minimax, then we have a

Saddle Point.

If there is a Saddle Point, then it is stable.
If both players know what the other will do, neither will change their strategy.

NOT EVERY GAME HAS A SADDLE POINT

	T_1	T_2	T_3	T_4	MIN	
S_1	5	6	20	3	**3**	
S_2	12	10	17	25	**10**	$\Leftarrow$
S_3	16	8	9	8	**8**	
S_4	13	18	6	5	**5**	
MAX	**16**	**18**	**20**	**25**		
	$\Uparrow$					

A Simpler Example

	T_1	T_2	Worst
S_1	8	-3	-3
S_2	-5	4	-5
Worst	8	4	

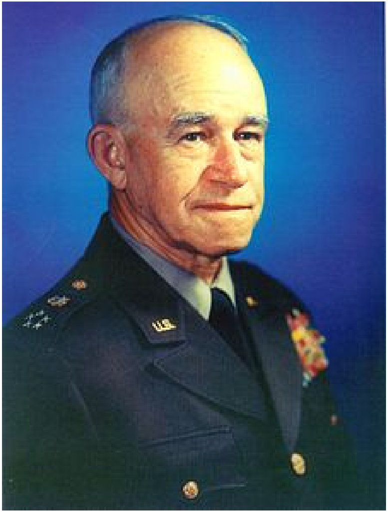
Here $\underline{v} = -3$ and $\bar{v} = 4$.
Rose can guarantee herself -3 by playing S_1.
Colin can limit her winnings to 4.
Rose tentatively selects S_1 and **Colin** selects T_2.
But if **Rose** knows **Colin** will play T_2, she should switch to S_2.
Knowing this, **Colin** switches to T_1 but.. Then **Rose** switches to S_1 causing **Colin** to...

An Historical Example

World War II

Click for More Information:

Operation Lüttich and Operation Tractable, August 1944



Omar Bradley
February 12, 1893 - April 8, 1981
More About Bradley

Günther Von Kluge
October 30, 1882 - August 19, 1944
More Abut Kluge

Example: In August 1944 after the invasion of Normandy, the Allies broke out of their beachhead at Avranches, France and headed into the main part of the country. The German General von Kluge, commander of the ninth army, faced two options:

- T_1: Stay and attack the advancing Allied armies.
- T_2: Withdraw into the mainland and regroup.

Simultaneously, General Bradley, commander of the Allied ground forces faced a similar set of options regarding the German ninth army:

- S_1 Reinforce the gap created by troop movements at Avranches
- S_2 Send his forces east to cut-off a German retreat
- S_3 Do nothing and wait a day to see what the adversary did.

In real life, there were no pay-off values, however General Bradley's diary indicates the scenarios he preferred in order. There are six possible scenarios. Bradley ordered them from most to least preferable and using this ranking, we can construct the game matrix.

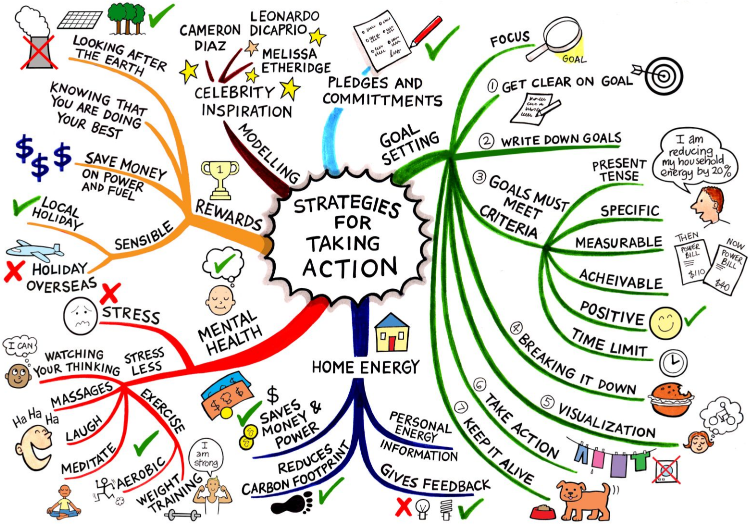
	Von Kluge's	**Strategies**	Row Min	
Bradley's Strategies	**Attack**	**Retreat**		
Reinforce Gap	2	3	2	
Move East	1	5	1	
Wait	6	4	4	$\Leftarrow \underline{v}$
Column Max	6	5		
		$\Uparrow \bar{v}$		

Notice that the maximin value of the rows is not equal to the minimax value of the columns. This is indicative of the fact that there is not a pair of strategies that form an equilibrium for this game.

	Von Kluge's	Strategies	Row Min	
Bradley's Strategies	**Attack**	**Retreat**		
Reinforce Gap	2	3	2	
Move East	1	5	1	
Wait	6	4	4	$\Leftarrow \underline{v}$
Column Max	6	5		
		$\Uparrow \bar{v}$		

Suppose that von Kluge plays his minimax strategy to retreat then Bradley would do better not to play his maximin strategy (wait) and instead move east, cutting of von Kluge's retreat, thus obtaining a payoff of (5, -5). But von Kluge would realize this and deduce that he should attack, which would yield a payoff of (1, -1). However, Bradley could deduce this as well and would know to play his maximin strategy (wait), which yields payoff (6, -6). However, von Kluge would realize that this would occur in which case he would decide to retreat yielding a payoff of (4, -4). The cycle then repeats.

How Should Each
Play The Game?
Choose a Strategy At Random!
But With What Probability?


STRATEGIES FOR TAKING ACTION
LOOKING AFTER THE EARTH
KNOWING THAT YOU ARE DOING YOUR BEST
SAVE MONEY ON POWER AND FUEL
LOCAL HOLIDAY
SENSIBLE
HOLIDAY OVERSEAS
REWARDS
CAMERON DIAZ
LEONARDO DICAPRIO
MELISSA ETHERIDGE
CELEBRITY INSPIRATION
MODELLING
PLEDGES AND COMMITTMENTS
GOAL SETTING
FOCUS
GOAL
① GET CLEAR ON GOAL
② WRITE DOWN GOALS
③ GOALS MUST MEET CRITERIA
PRESENT TENSE
I am reducing my household energy by 20%
SPECIFIC
MEASURABLE
THEN POWER BILL $110
NOW POWER BILL $40
ACHEIVABLE
POSITIVE
TIME LIMIT
④ BREAKING IT DOWN
⑤ VISUALIZATION
⑥ TAKE ACTION
⑦ KEEP IT ALIVE
MENTAL HEALTH
STRESS
I CAN
WATCHING YOUR THINKING
STRESS LESS
MASSAGES
EXERCISE
Ha Ha Ha
LAUGH
MEDITATE
AEROBIC
I am strong
WEIGHT TRAINING
HOME ENERGY
$
SAVES MONEY & POWER
REDUCES CARBON FOOTPRINT
PERSONAL ENERGY
INFORMATION
GIVES FEEDBACK

Mixed Strategies

Let p be the probability **Rose** plays S_1.

The $1 - p$ is the probability she plays S_2.

Similarly, q and $1 - q$ would represent the probabilities that **Colin** selects T_1 and T_2, respectively.

Then the probabilities of the various outcomes are given by

	q	$1 - q$
p	pq	$p(1 - q)$
$1 - p$	$(1 - p)q$	$(1 - p)(1 - q)$

with payoffs

	q	$1 - q$
p	8	-3
$1 - p$	-5	4

Expected Value for **Rose** =

$$\mathbf{EV} = 8pq - 3p(1 - q) + (-5)(1 - p)q + 4(1 - p)(1 - q)$$

Expected Value for **Rose** =

$$\mathbf{EV} = 8pq - 3p(1-q) + (-5)(1-p)q + 4(1-p)(1-q)$$

$$= 20pq - 7p - 9q + 4$$

$$= (4p - \frac{9}{5})(5q - \frac{7}{4}) - \frac{63}{4} + 4$$

$$= (4p - \frac{9}{5})(5q - \frac{7}{4}) + \frac{17}{20}$$

Rose: $4p = \frac{9}{5} \Rightarrow p = \frac{9}{20}$

Colin: $5q - \frac{7}{4} \Rightarrow q = \frac{7}{20}$

	T_1	T_2	Worst
S_1	8	-3	-3
S_2	-5	4	-5
Worst	8	4	

Here $\underline{v} = -3$ and $\bar{v} = 4$.

$$EV = (4p - \frac{9}{5})(5q - \frac{7}{4}) + \frac{17}{20}$$

With $p = \frac{9}{20}$ and $q = \frac{7}{20}$,
the Expected Value of the Game is $\frac{17}{20}$

$$\textbf{EV}_{\text{Rose}} = 20pq - 7p - 9q + 4$$

$$= (20q - 7)p - 9q + 4$$

So **Colin** should choose $q = \frac{7}{20}$

$$\textbf{EV}_{\text{Colin}} = -(20pq - 7p - 9q + 4)$$

$$= -20pq + 7p + 9q - 4 =$$

$$(9 - 20p)q + 7p - 4$$

So **Rose** should choose $p = \frac{9}{20}$

Another Way To Determine Equilibrium Strategies

Rose's View:
Mixed Strategy vs T_1 : $8p + (1-p)(-5) = 13p - 5$
Mixed Strategy vs T_2 : $-3p + 4(1-p) = 4 - 7p$

These payoffs are equal when

$$13p - 5 = 4 - 7p$$
$$20p = 9$$
$$p = \frac{9}{20}$$

Colin's Perspective:
Mixed Strategy vs S_1 : $8q + (1-q)(-3) = 11q - 3$
Mixed Strategy vs S_2 : $-5q + 4(1-q) = 4 - 9q$
These payoffs are equal when

$$11q - 3 = 4 - 9q \Rightarrow 20q = 7 \Rightarrow q = \frac{7}{20}$$

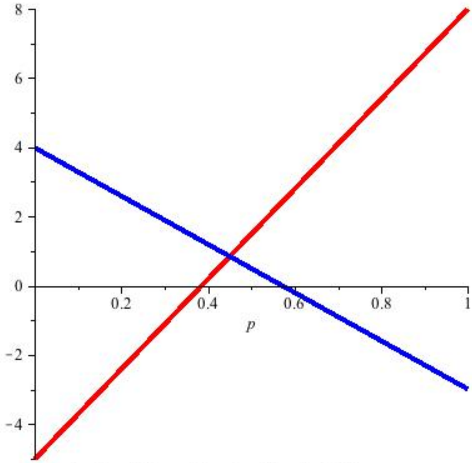
A Graphical Approach

Look at Rose's Expected Payoffs:

vs $T_1 : 13p - 5$

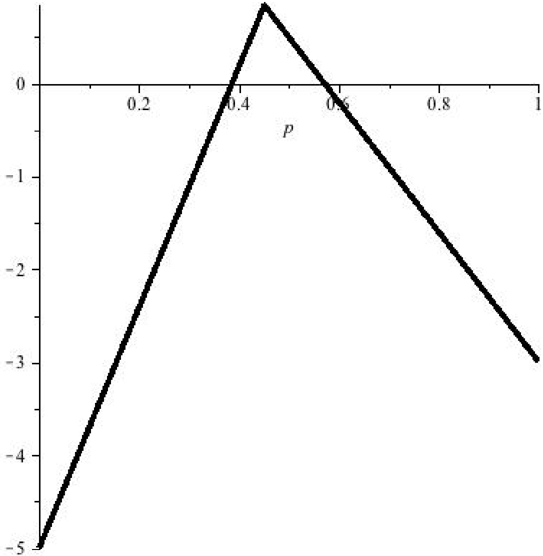
vs $T_2 : 4 - 7p$

Graph these on $0 \leq p \leq 1$.



A plot of $13\,p - 5$ in red and $4 - 7\,p$ in blue

Plot $\min(13p - 5, 4 - 7p)$ for $0 \leq p \leq 1$



Another Example

	T_1	T_2	T_3	T_4
S_1	1	0	3	- 3
S_2	-1	4	-2	6

	T_1	T_2	T_3	T_4	Row Minima
S_1	1	0	3	- 3	-**3**
S_2	-1	4	-2	6	-**2**
Column Maxima	**1**	**4**	**3**	**6**	

	T_1	T_2	T_3	T_4	Row Minima
S_1	1	0	3	- 3	-**3**
S_2	-1	4	-2	6	-**2** $\Leftarrow \underline{v}$
Column Maxima	**1** $\Uparrow$ $\bar{v} = 1$	4	3	6	

	T_1	T_2	T_3	T_4	Row Minima
S_1	1	0	3	- 3	**-3**
S_2	-1	4	-2	6	**-2** $\Leftarrow \underline{v}$
Column Maxima	**1** $\Uparrow$ $\bar{v} = 1$	4	3	6	

Value of this game is somewhere between -2 and 4.

Consider Expected Payoff to **Rose** if she uses S_1 with probability p and S_2 with probability $1 - p$.

vs $T_1 : 1p + (-1)(1-p) = 2p - 1$
vs $T_2 : 0p + 4(1-p) = 4 - 4p$
vs $T_3 : 3p + (-2)(1-p) = 5p - 2$
vs $T_4 : -3p + 6(1-p) = 6 - 9p$

Expected Payoff to **Rose** with mixture $(p, 1-p)$:

$$\begin{aligned} \text{vs } T_1 &: 2p - 1 \\ \text{vs } T_2 &: 4 - 4p \\ \text{vs } T_3 &: 5p - 2 \\ \text{vs } T_4 &: 6 - 9p \end{aligned}$$

Is there a single p which guarantees same expected payoff against all 4 of Colin's strategies?

T_1 and T_2: $2p - 1 = 4 - 4p \Rightarrow 6p = 5 \Rightarrow p = \frac{5}{6}$

With $p = \frac{5}{6}$:

Expected Value against $T_1 = 2(\frac{5}{6}) - 1 = 2/3$

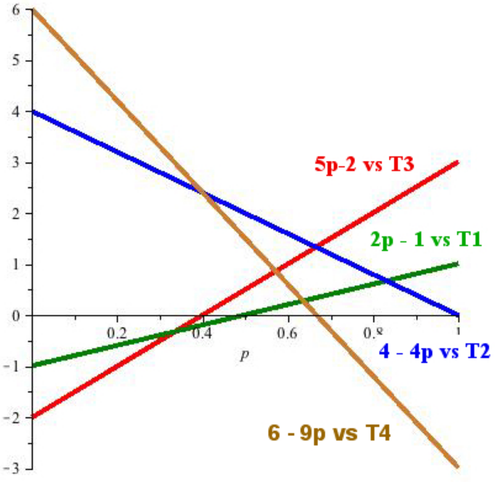
Expected Value against $T_2 = 4 - 4(\frac{5}{6}) = 2/3$

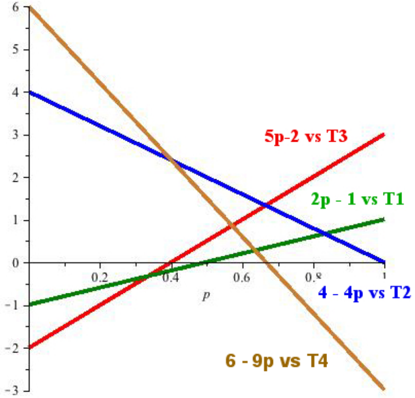
But

Expected Value against $T_3 = 5(\frac{5}{6}) - 2 = 13/6$

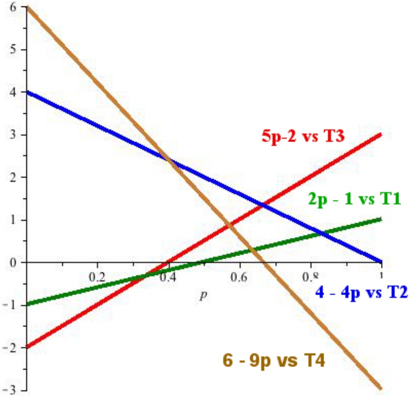
Expected Value against $T_4 = 6 - 9(\frac{5}{6}) = -3/2$

Graphical Approach



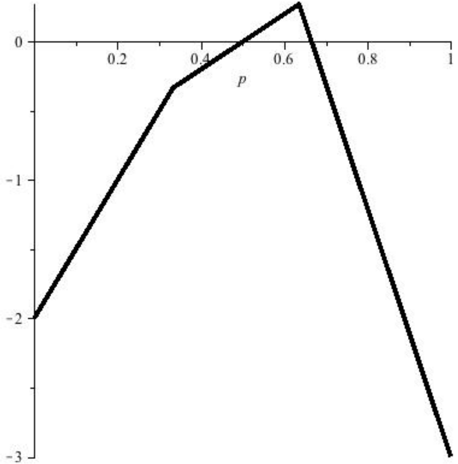


Look at bottom edge



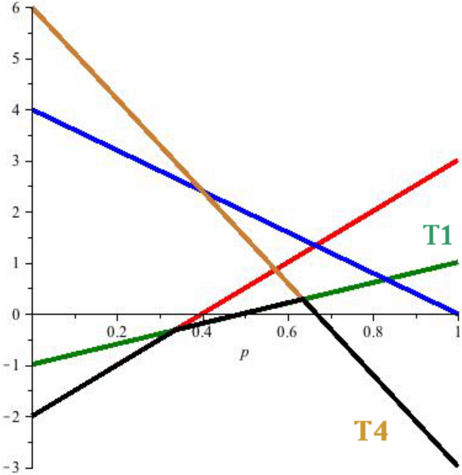
Look at bottom edge

Plot $\min(2p-1, 4-4p, 5p-2, 6-9p)$ for $0 \le p \le 1$



Rose can guarantee herself expected payoff of at least $\frac{3}{11}$ by choosing the strategy mixture $(\frac{7}{11}, \frac{4}{11})$.

Colin asks: Can I keep her winnings down to $\frac{3}{11}$?



6
5
4
3
2
1
0
-1
-2
-3
0.2
0.4
0.6
0.8
1
p
T1
T4

Colin wants to play a combination of T_1 and $T4$.

	q T_1	$1-q$ T_4
S_1	1	-3
S_2	-1	6

Expected Payoffs

vs S_1 : $1q - 3(1-q) = 4q - 3$

vs S_2 : $-1q + 6(1-q) = 6 - 7q$

We can solve for q by

Setting $4q - 3 = \frac{3}{11}$ or

Setting $6 - 7q = \frac{3}{11}$ or

Setting $4q - 3 = 6 - 7q$

All these lead to $q = \frac{9}{11}$

Colin's Optimal Mixture is $(\frac{9}{11}, 0, 0, \frac{2}{11})$

We Can Use Graphical Approach Whenever One of the Players Has Exactly 2 Strategies

Next Time:
Connecting Game Theory
With
Linear Programming