Game Theory II

Class 30

May 1, 2023

Advice on Team Project 2

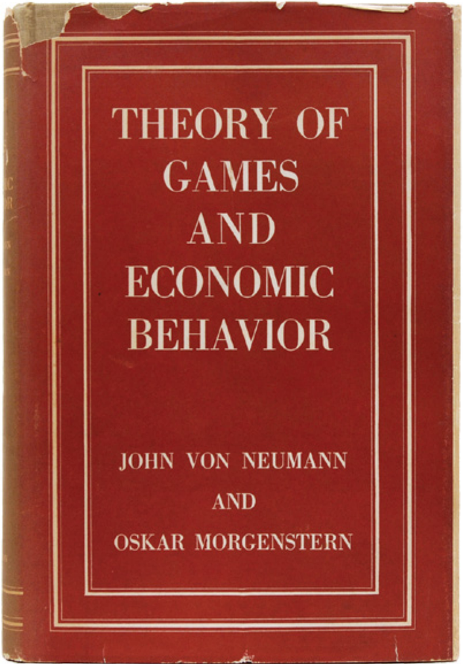
Some Advice on Assignment 11

On Problem 10.4-1:

- No Expected Values Here
- Want Probability he ends up with **exactly** 100 dollars
- Decision variable x_n = amount to bet on nth match
- States to consider: s_n = current fortune of the player before betting on the nth match.
- Interesting values of s to consider (perhaps): $0 \leq s < 25$, 25, $25 < s < 50$, 50, $50 < s < 75$, 75, $75 < s < 100$, 100, $s > 100$
- The only probabilities you should see are 0, 1/4, 1/2, 3/4 and 1.
- Worth looking carefully at solution to 10.4-2.
- It may be useful to look at a recursive relation of the form $f_n(s_n, x_n) = \frac{1}{2} f^*_{n+1}(s_n - x_n) + \frac{1}{2} f^*_{n+1}(s_n + x_n)$

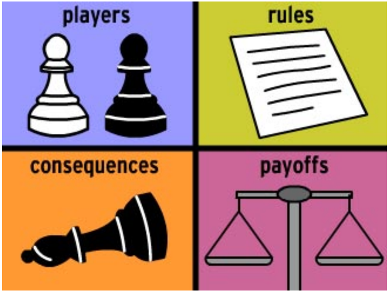
Introduction To The Theory of Games

Theory of Games and Economics Behavior



The Elephant in the Book

The Basic Ingredients of a Game



Players

Rules

Strategies

Outcomes

Strategies

- Infinite Number for Each Player
- **Finite Number for Each Player**

Focus on Strategy

Two-Person Game
With Finite Number of Strategies
m x n Game

- Outcome Matrix
- Payoff Matrix

Two-Person Game With Finite Number of Strategies
m x n Game

- Outcome Matrix
- Payoff Matrix

Example: 2 x 3 Games Between **Rose** and **Colin**

Rose has 2 strategies (S_1, S_2)

Colin has 3 strategies (T_1, T_2, T_3)

Outcome Matrix

		Colin		
		T_1	T_2	T_3
Rose	S_1	O_{11}	O_{12}	O_{13}
	S_2	O_{21}	O_{22}	O_{23}

Outcome Matrix

		Colin		
		T_1	T_2	T_3
Rose	S_1	O_{11}	O_{12}	O_{13}
	S_2	O_{21}	O_{22}	O_{23}

Payoff Matrix

		Colin		
		T_1	T_2	T_3
Rose	S_1	(a_1, b_1)	(a_2, b_2)	(a_3, b_3)
	S_2	(a_4, b_4)	(a_5, b_5)	(a_6, b_6)

Zero-Sum Game

$b_i = -a_i$ for all i

Display only payoff to Row Player

		Colin		
		T_1	T_2	T_3
	S_1	1	2	0
Rose	S_2	2	-3	-2
	S_3	0	3	-1

Bigger Numbers are Better For **Rose**
Smaller Numbers are Better for **Colin**

Dominating Strategy

		Colin		
		T_1	T_2	T_3
	S_1	1	2	0
Rose	S_2	2	-3	-2
	S_3	0	3	-1

$T_3 > T_1$: **Colin** will never play $T1$

Game Becomes

		Colin	
		T_2	T_3
	S_1	2	0
Rose	S_2	-3	-2
	S_3	3	-1

		Colin	
		T_2	T_3
	S_1	2	0
Rose	S_2	-3	-2
	S_3	3	-1

But now $S_1 > S_2$ so **Rose** will never select S_2

		Colin	
		T_2	T_3
Rose	S_1	2	0
	S_3	3	-1

		Colin	
		T_2	T_3
Rose	S_1	2	0
	S_3	3	-1

Now $T_3 > T_2$ so eliminate T_2

		Colin
		T_3
Rose	S_1	0
	S_3	-1

Finally, $S_1 > S_2$ so eliminate S_2:

		Colin
		T_3
Rose	S_1	0

Eliminate Dominated Strategies If They Exist

Finding Your Optimal Strategy in Zero-Sum Game

- Eliminate All Dominated Strategies
- Determine "Best of the Worst"

Here's Another Way To Analyze This Game:

	T_1	T_2	T_3	Worst for **Rose**
S_1	1	2	0	**0**
S_2	2	-3	-2	**-3**
S_3	0	3	-1	**-1**
Worst for **Colin**	**2**	**3**	**0**	

Choose Best of Worst
Maximin or Minimax

	T_1	T_2	T_3	T_4	MIN	
S_1	5	6	20	3	**3**	
S_2	12	$\boxed{10}$	17	25	**10**	$\Leftarrow$
S_3	16	8	9	8	**8**	
S_4	13	9	6	5	**5**	
MAX	**16**	**10**	**20**	**25**		
		$\Uparrow$				

maximin $= \underline{v}$ = lower value

minimax $= \bar{v}$ = upper value

If maximin = minimax, then we have a

Saddle Point.

If there is a Saddle Point, then it is stable.
If both players know what the other will do, neither will change their strategy.

"Best of Worst"

Maximin

For Rose: Find Minimum in Each Row

Take Maximum of these Minima

Lower Value $\underline{v}$ for Game


MAXIMIN


Nature Made®
DAILY
Maximin™ Pack
Maximum Strength
For Men and Women
with LUTEIN
FOR EYE HEALTH†
6 SUPPLEMENTS PER PACKET
Balanced B-100
Vitamin C
Vitamin E
Calcium
Vitamin A & D
Multivitamin
30 PACKETS | 30 DAY SUPPLY
DIETARY SUPPLEMENT
Supplement Facts

"Best of Worst"

Minimax

For Colin: Find Maximum in Each Column

Take Minimum of these Maxima

Upper Value $\overline{v}$ for Game

MINIMAX

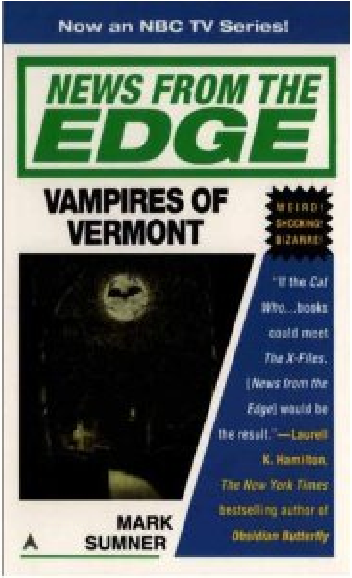

NEW
Kellogg's
mini
MAX
the wheat that goes a long way
ORIGINAL
High Fibre
Wholegrain

"Best of Worst"
Maximin and Minimax Strategies
Lower Value and Upper Value
Saddle Points

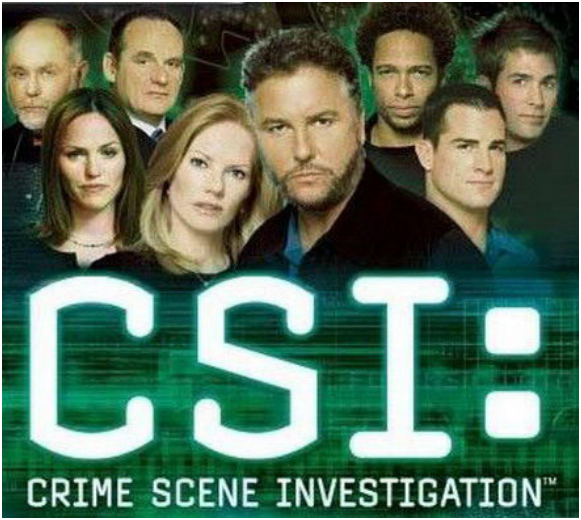
Two television networks believe there are 100 million potential viewers for Thursday night, prime-time (8 - 9 PM).

The networks must decide which type of program to broadcast: Science Fiction (*Vermont Vampires*), Drama(*CSI Middlebury*), or Comedy (*Old Chapel's Big Bang Theory*).

If the two networks initially split the 100 million viewers evenly, think of the payoff matrix entries as how many excess viewers the network's strategies will yield over 50 million.



Vermont Vampires



CSI Middlebury

Old Chapel's Big Bang Theory

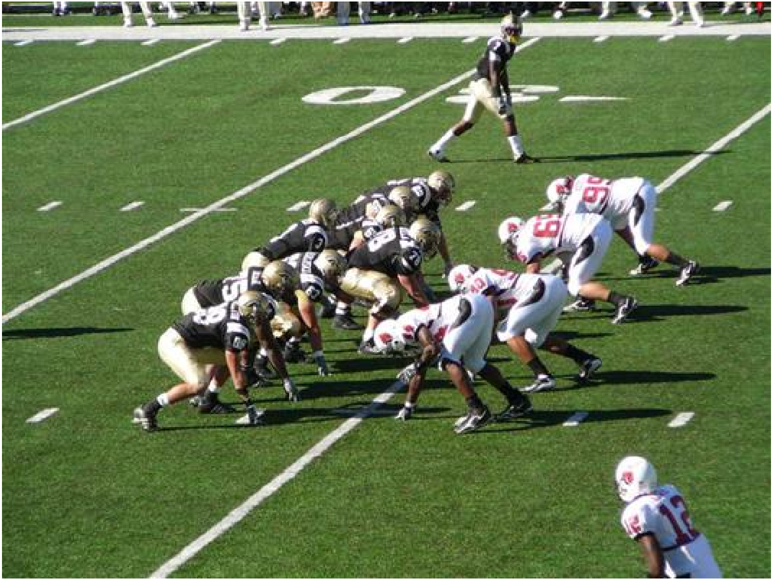
	SciFi	Drama	Comedy	Row's Worst	
SciFic	-15	-35	10	**-35**	
Drama	-5	8	0	**-5**	$\Leftarrow$
Comedy	-12	-36	20	**-36**	
Column's Worst	**-5**	**8**	**20**		
	$\Uparrow$				

Choose Best of Worst
Maximin or Minimax

NOT EVERY GAME HAS A SADDLE POINT

	T_1	T_2	T_3	T_4	MIN	
S_1	5	6	20	3	**3**	
S_2	12	10	17	25	**10**	$\Leftarrow$
S_3	16	8	9	8	**8**	
S_4	13	18	6	5	**5**	
MAX	**16**	**18**	**20**	**25**		
	$\Uparrow$					


PANTHERS
MIDDLEBURY



A football play (in which the scores does not change) is an example of a zero-sum game when the payoff is measured by yards gained or lost. In a football game, there are two players: Offense and Defense.

For simplicity, suppose the Offense has two strategies: Pass or Run while the Defense has three strategies: Pass Defense, Run Defense, or Blitz.

	Pass Defense	Run Defense	Blitz	Row's Worst	
Pass	-3	9	-5	**-5**	
Run	4	-3	6	**-3**	⇐
Column's Worst	**4**	**9**	**6**		
	⇑				

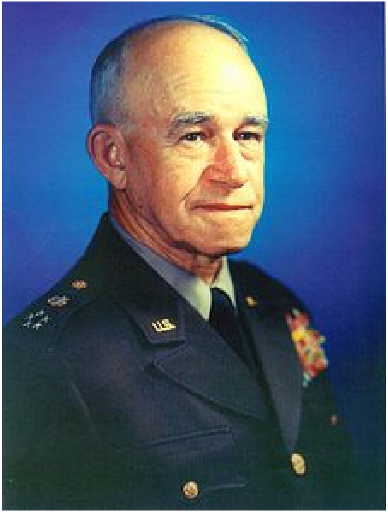
There is No Saddle Point

An Historical Example

World War II

Click for More Information:

Operation Lüttich and Operation Tractable, August 1944



Omar Bardley

Günther Von Kluge

Example: In August 1944 after the invasion of Normandy, the Allies broke out of their beachhead at Avranches, France and headed into the main part of the country. The German General von Kluge, commander of the ninth army, faced two options:

- T_1: Stay and attack the advancing Allied armies.
- T_2: Withdraw into the mainland and regroup.

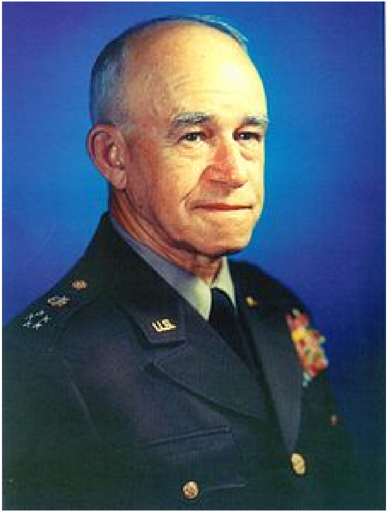
Simultaneously, General Bradley, commander of the Allied ground forces faced a similar set of options regarding the German ninth army:

- S_1 Reinforce the gap created by troop movements at Avranches
- S_2 Send his forces east to cut-off a German retreat
- S_3 Do nothing and wait a day to see what the adversary did.

In real life, there were no pay-off values, however General Bradley's diary indicates the scenarios he preferred in order. There are six possible scenarios. Bradley ordered them from most to least preferable and using this ranking, we can construct the game matrix.

	Von Kluge's	**Strategies**	Row Min	
Bradley's Strategies	**Attack**	**Retreat**		
Reinforce Gap	2	3	2	
Move East	1	5	1	
Wait	6	4	4	$\Leftarrow \underline{v}$
Column Max	6	5		
		$\Uparrow \bar{v}$		

Notice that the maximin value of the rows is not equal to the minimax value of the columns. This is indicative of the fact that there is not a pair of strategies that form an equilibrium for this game.



Omar Bardley

Günther Von Kluge

	Von Kluge's	Strategies	Row Min	
Bradley's Strategies	**Attack**	**Retreat**		
Reinforce Gap	2	3	2	
Move East	1	5	1	
Wait	6	4	4	$\Leftarrow \underline{v}$
Column Max	6	5		
		$\Uparrow \bar{v}$		

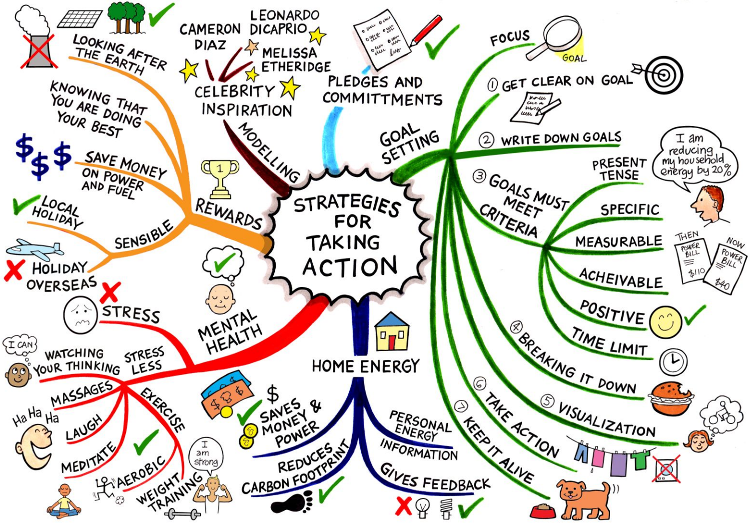
Suppose that von Kluge plays his minimax strategy to retreat then Bradley would do better not to play his maximin strategy (wait) and instead move east, cutting of von Kluge's retreat, thus obtaining a payoff of (5, -5). But von Kluge would realize this and deduce that he should attack, which would yield a payoff of (1, -1). However, Bradley could deduce this as well and would know to play his maximin strategy (wait), which yields payoff (6, -6). However, von Kluge would realize that this would occur in which case he would decide to retreat yielding a payoff of (4, -4). The cycle then repeats.

A Simpler Example

	T_1	T_2	Worst
S_1	8	-3	-3
S_2	-5	4	-5
Worst	8	4	

Here $\underline{v} = -3$ and $\bar{v} = 4$.
Rose can guarantee herself -3 by playing S_1.
Colin can limit her winnings to 4.
Rose tentatively selects S_1 and **Colin** selects T_2.
But if **Rose** knows **Colin** will play T_2, she should switch to S_2.
Knowing this, **Colin** switches to T_1 but.. Then **Rose** switches to S_1 causing **Colin** to...

How Should Each
Play The Game?
Choose a Strategy At Random!
But With What Probability?


STRATEGIES FOR TAKING ACTION
GOAL SETTING
FOCUS
GOAL
① GET CLEAR ON GOAL
② WRITE DOWN GOALS
③ GOALS MUST MEET CRITERIA
PRESENT TENSE
I am reducing my household energy by 20%
SPECIFIC
MEASURABLE
THEN POWER BILL $110
NOW POWER BILL $40
ACHEIVABLE
POSITIVE
TIME LIMIT
④ BREAKING IT DOWN
⑤ VISUALIZATION
⑥ TAKE ACTION
⑦ KEEP IT ALIVE
PLEDGES AND COMMITTMENTS
MODELLING
CELEBRITY INSPIRATION
CAMERON DIAZ
LEONARDO DICAPRIO
MELISSA ETHERIDGE
REWARDS
LOOKING AFTER THE EARTH
KNOWING THAT YOU ARE DOING YOUR BEST
SAVE MONEY ON POWER AND FUEL
SENSIBLE
LOCAL HOLIDAY
HOLIDAY OVERSEAS
MENTAL HEALTH
STRESS
STRESS LESS
I CAN
WATCHING YOUR THINKING
MASSAGES
LAUGH
Ha Ha Ha
MEDITATE
EXERCISE
AEROBIC
WEIGHT TRAINING
I am strong
HOME ENERGY
$ SAVES MONEY & POWER
REDUCES CARBON FOOTPRINT
PERSONAL ENERGY INFORMATION
GIVES FEEDBACK

Mixed Strategies

Let p be the probability **Rose** plays S_1.
The $1-p$ is the probability she plays S_2.
Similarly, q and $1-q$ would represent the probabilities that **Colin** selects T_1 and T_2, respectively.
Then the probabilities of the various outcomes are given by

	q	$1-q$
p	pq	$p(1-q)$
$1-p$	$(1-p)q$	$(1-p)(1-q)$

with payoffs

	q	$1-q$
p	8	-3
$1-p$	-5	4

Expected Value for **Rose** =

$$\mathbf{EV} = 8pq - 3p(1-q) + (-5)(1-p)q + 4(1-p)(1-q)$$

Expected Value for **Rose** =

$$\textbf{EV} = 8pq - 3p(1-q) + (-5)(1-p)q + 4(1-p)(1-q)$$

$$= 20pq - 7p - 9q + 4$$

$$= (4p - \frac{9}{5})(5q - \frac{7}{4}) - \frac{63}{4} + 4$$

$$= (4p - \frac{9}{5})(5q - \frac{7}{4}) + \frac{17}{20}$$

Rose: $4p = \frac{9}{5} \Rightarrow p = \frac{9}{20}$

Colin: $5q - \frac{7}{4} \Rightarrow q = \frac{7}{20}$

	T_1	T_2	Worst
S_1	8	-3	-3
S_2	-5	4	-5
Worst	8	4	

Here $\underline{v} = -3$ and $\bar{v} = 4$.

$$EV = (4p - \frac{9}{5})(5q - \frac{7}{4}) + \frac{17}{20}$$

With $p = \frac{9}{20}$ and $q = \frac{7}{20}$,
the Expected Value of the Game is $\frac{17}{20}$

$$\mathbf{EV}_{\text{Rose}} = 20pq - 7p - 9q + 4$$

$$= (20q - 7)p - 9q + 4$$

So **Colin** should choose $q = \frac{7}{20}$

$$\mathbf{EV}_{\text{Colin}} = -(20pq - 7p - 9q + 4)$$

$$= -20pq + 7p + 9q - 4 =$$

$$(9 - 20p)q + 7p - 4$$

So **Rose** should choose $p = \frac{9}{20}$