# Dynamic Programming IV
# Introduction to Game Theory


Class 29

April 28, 2023

# Assignment 10
# Due Wednesday

**Notes on Exam 2**

**Notes on Assignment 9**

**Assignment 11**

*Some Advice*

*on Assignment 11*

# Fromage Cheese Problem Via Dynamic Programming

$$\text{Maximize } 4.5x + 4y$$

$$\text{subject to}$$

$$30x + 12y \leq 6000 (C = cheddar)$$

$$10x + 8y \leq 2600 \ (S = Swiss)$$

$$4x + 8y \leq 2000 (B = Brie)$$

$(C, S, B)$ measures slacks

Stage 1: Choose $x$

Stage 2: Choose $y$

States: Remaining amounts of resources $(C, S, B)$

For $n = 2$, $f_2^*(C, S, B) = \max 4y$

where

$$\begin{array}{ccc} 12y \le C & & y \le C/12 \\ 8y \le S & \textit{implies} & y \le S/8 \\ 8y \le B & & y \le B/8 \end{array}$$

| $(C, S, B)$ | $f_2^*(C, S, B)$ |
| --- | --- |
| all $\ge 0$ | $4 \min\left(\frac{C}{12}, \frac{S}{8}, \frac{B}{8}\right)$ |

$$f_2^*(C,S,B) = 4\min\left(\frac{C}{12}, \frac{S}{8}, \frac{B}{8}\right)$$

Now for $n=1$: $f_1((6000,2600,2000),x) =$

$$\max\left(4.5x + 4\min\left(\frac{6000-30x}{12}, \frac{2600-10x}{8}, \frac{2000-4x}{8}\right)\right)$$

$$= \max\left(4.5x + 4\min\left(\frac{1000-5x}{2}, \frac{1300-5x}{4}, \frac{500-x}{2}\right)\right)$$

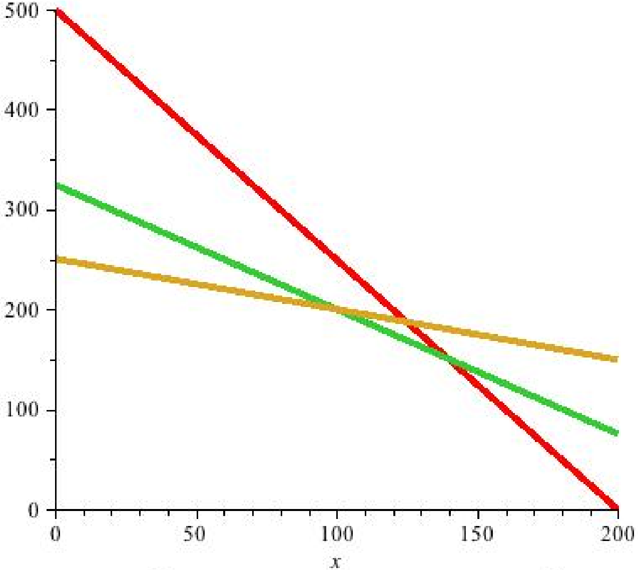
$$= \max_{0\le x\le 200}\left(4.5x + 4\min\left(\frac{1000-5x}{2}, \frac{1300-5x}{4}, \frac{500-x}{2}\right)\right)$$

Now

$$\min\left(\frac{1000-5x}{2}, \frac{1300-5x}{4}, \frac{500-x}{2}\right)$$

$$=$$

$$= \begin{cases} \frac{500-x}{2} = 250 - \frac{x}{2} & \text{if } 0 \leq x \leq 100 \\ \frac{1300-5x}{4} = 325 - \frac{5}{4}x & \text{if } 100 \leq x \leq 140 \\ \frac{1000-5x}{2} = 500 - \frac{5}{2}x & \text{if } 140 \leq x \leq 200 \end{cases}$$




A plot of $500 - \frac{5}{2}x$ in red, $325 - \frac{5}{4}x$ in green and $250 - \frac{1}{2}x$ in gold

Thus

$$f_1^* = max \begin{cases} 4.5x + 2(500 - x) = 2.5x + 1000 & \text{if } 0 \leq x \leq 100 \\ 4.5x + (1300 - 5x) = 1300 - .5x & \text{if } 100 \leq x \leq 140 \\ 4.5x + (2000 - 10x) = 2000 - 5.5x & \text{if } 140 \leq x \leq 200 \end{cases}$$

$$f_1^* = max \begin{cases} 2.5 \cdot 100 + 1000 = 1250 & \text{if } 0 \leq x \leq 100 \\ 1300 - .5 \cdot 100 = 1250 & \text{if } 100 \leq x \leq 140 \\ 2000 - 5.5 \cdot 140 = 1230 & \text{if } 140 \leq x \leq 200 \end{cases}$$

Thus $f_1^* = 1250$ at $x = 100$.

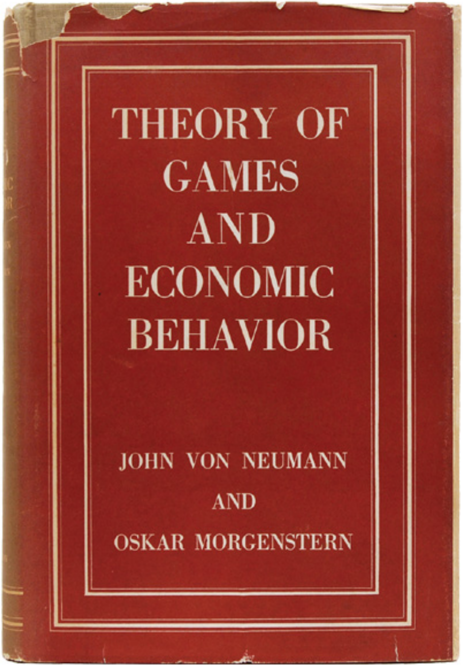
Then

$$\begin{array}{lcl} C = 6000 - 3000 = 3000 & & C/12 = 250 \\ S = 2600 - 1000 = 1600 & \text{yields} & S/8 = 200 \\ B = 2000 - 400 = 1600 & & B/8 = 200 \end{array}$$

Minimum $= 200$ so $y = 200$.

Hence the optional solution is $x = 100, y = 200$ with a Revenue of \$ 1250.

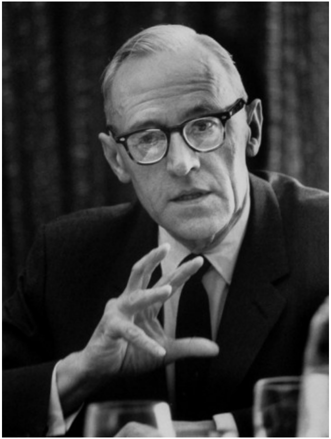
# Introduction To The Theory of Games


THEORY OF GAMES AND ECONOMIC BEHAVIOR
JOHN VON NEUMANN
AND
OSKAR MORGENSTERN

# John Von Neumann

Born: December 28, 1903 in Budapest, Hungary
Died: February 8, 1957 in Washington D.C.

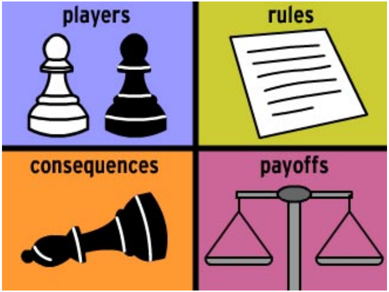
# Oskar Morgenstern



Born: January 24, 1902 in Görlitz, Germany
Died July 26, 1977 in Princeton, New Jersey.

**Why Classical Mathematics Was Not Sufficient**

# The Basic Ingredients of a Game




Players

Rules

Strategies

Outcomes

# Number of Players

1-person games (Decision Theory)

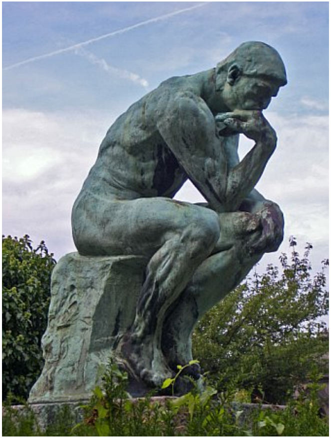
2-person games

$n$-person games (Coalitions)

Infinitely many people (atomic games)

# Fundamental Assumption About Players

They Act Rationally
All Are Equally Smart
Each Tries To Maximize Utility



# Rules

Chance

Perfect or Imperfect Information

Communication

# **Moves** vs **Strategies**

Penny Removal Game

**Move**: Take away 1,2, or 3 pennies

**Strategies**:

Always Take Away 2

Remove what your opponent does

Take 4-$x$ if your opponent takes $x$

## Strategies

- Infinite Number for Each Player
- Finite Number for Each Player

# Two-Person Game With Finite Number of Strategies
# $m$ x $n$ Game

- Outcome Matrix
- Payoff Matrix

# Example: 2 x 3 Games Between **Rose** and **Colin**

**Rose** has 2 strategies ($S_1$, $S_2$)

**Colin** has 3 strategies ($T_1$, $T_2$, $T_3$)

Outcome Matrix

| | | **Colin** | | |
|---|---|---|---|---|
| | | $T_1$ | $T_2$ | $T_3$ |
| **Rose** | $S_1$ | $O_{11}$ | $O_{12}$ | $O_{13}$ |
| | $S_2$ | $O_{21}$ | $O_{22}$ | $O_{23}$ |

## Outcome Matrix

| | | Colin | | |
|---|---|---|---|---|
| | | $T_1$ | $T_2$ | $T_3$ |
| Rose | $S_1$ | $O_{11}$ | $O_{12}$ | $O_{13}$ |
| | $S_2$ | $O_{21}$ | $O_{22}$ | $O_{23}$ |

## Payoff Matrix

| | | Colin | | |
|---|---|---|---|---|
| | | $T_1$ | $T_2$ | $T_3$ |
| Rose | $S_1$ | $(a_1, b_1)$ | $(a_2, b_2)$ | $(a_3, b_3)$ |
| | $S_2$ | $(a_4, b_4)$ | $(a_5, b_5)$ | $(a_6, b_6)$ |

# Zero-Sum Game

$b_i = -a_i$ for all $i$

Display only payoff to Row Player

| | | Colin | | |
|---|---|---|---|---|
| | | $T_1$ | $T_2$ | $T_3$ |
| | $S_1$ | 1 | 2 | 0 |
| Rose | $S_2$ | 2 | -3 | -2 |
| | $S_3$ | 0 | 3 | -1 |

Bigger Numbers are Better For **Rose** Smaller Numbers are Better for **Colin**

# Dominating Strategy

| | | Colin | | |
|---|---|---|---|---|
| | | $T_1$ | $T_2$ | $T_3$ |
| | $S_1$ | 1 | 2 | 0 |
| Rose | $S_2$ | 2 | -3 | -2 |
| | $S_3$ | 0 | 3 | -1 |

$T_3 > T_1$: **Colin** will never play $T1$

Game Becomes

| | | Colin | |
|---|---|---|---|
| | | $T_2$ | $T_3$ |
| | $S_1$ | 2 | 0 |
| Rose | $S_2$ | -3 | -2 |
| | $S_3$ | 3 | -1 |

| | | Colin | |
|---|---|---|---|
| | | $T_2$ | $T_3$ |
| | $S_1$ | 2 | 0 |
| Rose | $S_2$ | -3 | -2 |
| | $S_3$ | 3 | -1 |

But now $S_1 > S_2$ so **Rose** will never select $S_2$

| | | Colin | |
|---|---|---|---|
| | | $T_2$ | $T_3$ |
| Rose | $S_1$ | 2 | 0 |
| | $S_3$ | 3 | -1 |

| | | Colin | |
|---|---|---|---|
| | | $T_2$ | $T_3$ |
| Rose | $S_1$ | 2 | 0 |
| | $S_3$ | 3 | -1 |

**Now $T_3 > T_2$ so eliminate $T_2$**

| | | Colin |
|---|---|---|
| | | $T_3$ |
| Rose | $S_1$ | 0 |
| | $S_3$ | -1 |

**Finally, $S_1 > S_2$ so eliminate $S_2$:**

| | | Colin |
|---|---|---|
| | | $T_3$ |
| Rose | $S_1$ | 0 |

**Eliminate Dominated Strategies If They Exist**