Duality: Round 3

Class 14

Wednesday, March 15, 2023

Exam 1
Tonight Starting at 7 PM
Here

Comments About Assignment 5

- Weak Duality Theorem Will Be Helpful
- Problems with No ≥ 0 Constraints on Decision Variables
  Let $x = x^+ - x^-$.
  Suppose optimal solution has $x^+ = 4$ and $x^- = 7$.
  Then $x = ?$
- What is meant by Primal-Dual Table?

	x_1	x_2	
y_1	30	12	≤ 6000
y_2	10	8	≤ 2600
y_3	4	8	≤ 2000
	IV	IV	
	4.5	4	

Introduction
To
Duality Part III

Weak Duality Theorem

If $\mathbf{x}$ is a feasible solution to the Primal Problem and $\mathbf{y}$ is a feasible solution of the Dual Problem, then $\mathbf{cx} \leq \mathbf{yb}$.

Proof of Weak Duality Theorem

Suppose $\mathbf{x}$ is a feasible solution to the Primal Problem and $\mathbf{y}$ is a feasible solution of the Dual Poblem

Primal Problem	*Dual Problem*
Maximize $Z = c\mathbf{x}$	Minimize $W = \mathbf{y}b$
subject to	subject to
$A\mathbf{x} \leq b$	$\mathbf{y}A \geq c$
and $\mathbf{x} \geq 0$	and $\mathbf{y} \geq 0$.

Suppose $\mathbf{x}$ is a feasible solution to the Primal Problem and $\mathbf{y}$ is a feasible solution of the Dual Poblem
Begin with $\mathbf{c} \leq \mathbf{y}A$ and $A\mathbf{x} \leq \mathbf{b}$.
Multiply the first inequality by $\mathbf{x}$ on the right.
Multiply the second inequality by $\mathbf{y}$ on the left.

Then

$$\mathbf{cx} \leq (\mathbf{y}A)\mathbf{x} = \mathbf{y}(A\mathbf{x}) \leq \mathbf{yb}.$$

Weak Duality Theorem

If $\mathbf{x}$ is a feasible solution to the primal problem and $\mathbf{y}$ is a feasible solution of the dual problem, then $\mathbf{cx} \leq \mathbf{yb}$.

- Corollary 1: Any feasible solution of the dual gives a bound for the primal.
- Corollary 2: Any feasible solution of the primal gives a bound for the dual.
- Corollary 3: If the primal is unbounded, then the dual is infeasible.
- Corollary 4: If primal and dual both have feasible solutions, then both have optimal solutions.
- Corollary 5: Suppose $\mathbf{x}$ is feasible for primal and $\mathbf{y}$ is feasible for dual. If $\mathbf{cx} = \mathbf{yb}$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions.

Srong Duality Theorem

Strong Duality Property: If $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then $\mathbf{cx}^* = \mathbf{yb}^*$.

At each iteration, the simplex method simultaneously identifies a CPF solution $\mathbf{x}$ for the primal problem and a complementary solution $\mathbf{y}$ for the dual problem (in objective function row as the coefficients of the slack variables) where $\mathbf{cx} = \mathbf{yb}$. If $\mathbf{x}$ is not optimal for the primary problem, then $\mathbf{y}$ is not feasible for the dual problem.

Complementary Optimal Solutions Property: At the final iteration, the simplex method simultaneously identifies an optimal solution $\mathbf{x^*}$ for the primal problem and a complementary optimal solution $\mathbf{y^*}$ for the dual problem where $\mathbf{cx^*} = \mathbf{y^*b}$.

The components of $\mathbf{y}$ are the shadow prices for the primal problem.

Symmetry Property: For any primal problem and its dual problem, all relationships between them must be symmetric because the dual of this dual problem is the primal problem.

Duality Theorem

The following are the only possible relationships between the primal and dual problems:

- If one problem has feasible solutions and a bounded objective function (and therefore has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
- If one problem has feasible solutions and an unbounded objective function (and hence no optimal solution), then the other problem has no feasible solutions.
- If one problem has no feasible solutions, then the other problem has no feasible solutions or an unbounded objective function.

Shadow Prices

At each iteration the value of the objective function is given by

$$Z = \mathbf{cx} = \mathbf{yb} = b_1 y_1 + b_2 y_2 + .. + b_m y_m$$

We can interpret $y_i b_i$ as the current contribution to the objective function by having b_i units of resource i available for the primal.
Thus y_i is the contribution to objective function per unit of resource i when current set of basic variables is used to obtain the primal solution.

Final Tableaux

Tableau for the Optimal Basic Feasible Solution of Primal

	Z	x	y	u	v	w	
Z	1	0	0	0	5/12	1/12	1250
x	0	1	0	0	1/6	-1/6	100
y	0	0	1	0	-1/12	5/24	200
u	0	0	0	1	-4	5/2	600

Tableau for the Optimal Basic Feasible Solution of the Dual

	Z	C	S	B	S1	S2	
Z	1	600	0	0	100	200	-1250
S	0	4	1	0	-1/6	1/12	5/12
B	0	-5/2	0	1	1/6	-5/24	1/12

Another Way To Look at Duality

Fromage Cheese Problem

$$\text{Maximize } Z = 4.5x + 4y$$

subject to

$$30x + 12y \leq 6000 \text{ (Cheddar)}$$
$$10x + 8y \leq 2600 \text{ (Swiss)}$$
$$4x + 8y \leq 2000 \text{ (Brie)}$$
$$x, y \geq 0$$

Suppose we want to find an upper bound for Z

Multiply Cheddar constraint by $\frac{1}{6}$: $5x + 2y \leq 1000$

Multiply Brie constraint by $\frac{1}{4}$: $x + 2y \leq 500$

Now Add: $6x + 4y \leq 1500$

Then $Z = 4.5x + 4y \leq 6x + 4y \leq 1500$

Is there a **best** set of multipliers (C,S,B)?

Best Multipliers for Best Upper Bound

We want non-negative numbers C, S, B so that
$30Cx + 12Cy \leq 6000C$
$10Sx + 8Sy \leq 2600S$
$4Bx + 8By \leq 2000B$
Add:
$(30C+10S+4B)x+(12C+8S+8B)y \leq 6000C+2600S+2000B$

We need:
$4.5x \leq (30C + 10S + 4B)x$
$4y \leq (12C + 8S + 8B)y$
and
$6000C + 2600S + 2000B$ as small as possible.