Duality and The Fundamental Insight

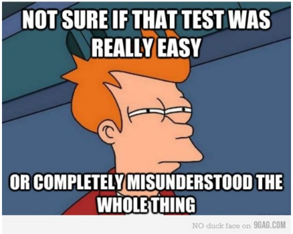
Class 12

March 10, 2023

Handouts

- Cycling in Simplex Method
- Assignment 5

Announcements



Exam 1

Wednesday at 7 PM

Warner 101

Exam 1 Details

No Time Limit

Show All Work

Double Check Your Answers

Justify Claims

Closed Book

No Calculator

What Are You "Responsible" For?

All Readings:

- Chapters 1 -4 of Hillier and Lieberman
- Olinick LP Notes
- Cooper, Bhat and LeBlanc

Everything

- I've Said
- I Should Have Said But Didn't
- I Might Have Said
- The State of the World


Any questions?

What Questions Remain?

Examine Final Tableaux of Fromage and Cheese Buyer Problems:

	Z	x	y	u	v	w	
Z	1	0	0	0	**5/12**	**1/12**	1250
x	0	1	0	0	**1/6**	**-1/6**	100
y	0	0	1	0	**-1/12**	**5/24**	200
u	0	0	0	1	**-4**	**5/2**	600

What is the meaning of the green numbers?

	Z	x	y	z	u	v	
Z	1	**600**	0	0	**100**	**200**	-1250
y	0	**4**	1	0	**-1/6**	**1/12**	**5/12**
z	0	**-5/2**	0	1	**1/6**	- **-5/24**	**1/12**

Duality And The Fundamental Insight

Fromage Cheese Problem	*Dual Problem*
Maximize $4.5x + 4y$ subject to $30x + 12y \leq 6000$ (Cheddar) $10x + 8y \leq 2600$ (Swiss) $4x + 8y \leq 2000$ (Brie)	Minimize $6000C + 2600S + 2000B$ subject to $30C + 10S + 4B \geq 4.5$ $12C + 8S + 8B \geq 4$
$x, y \geq 0$	$C, S, B \geq 0$
x and y are number of packages of each assortment to prepare	C, S, B are the price per ounce to offer for the cheeses

	x_1	x_2	
y_1	30	12	≤ 6000
y_2	10	8	≤ 2600
y_3	4	8	≤ 2000
	IV	IV	
	4.5	4	

Why Bother With Duality?

- Gain more insight into information the Simplex Method provides in the final tableau
- Run time (number of iterations) of Simplex Method depends more on the number of constraints than the number of decision variables. It may be quicker to solve the dual problem.

Example: Consider an LP with 10 variables and 5 $\leq$ inequality constraints.
Convert to equations by introducing 5 slack variables.

We have a system of 5 equations in $10 + 5 = 15$ variables.
The number of potential bases is $\binom{15}{5} = 3003$.

(a) Add 5 variables: 15 variables. 5 inequalities.
Number of potential bases is $\binom{20}{5} = 15,504$.

(b) Add 5 new inequalities so we have 10 variables, 10 $\leq$ constraints.
Number of potential bases is $\binom{20}{10} = 184,756$.

Final Tableaux

Tableau for the Optimal Basic Feasible Solution of Primal

	Z	x	y	u	v	w	
Z	1	0	0	0	5/12	1/12	1250
x	0	1	0	0	1/6	-1/6	100
y	0	0	1	0	-1/12	5/24	200
u	0	0	0	1	-4	5/2	600

Tableau for the Optimal Basic Feasible Solution of the Dual

	Z	C	S	B	S1	S2	
Z	1	600	0	0	100	200	-1250
S	0	4	1	0	-1/6	1/12	5/12
B	0	-5/2	0	1	1/6	-5/24	1/12

Fundamental Insight: Green Numbers Record Row Operations

Initial Tableau

	Z	x	y	u	v	w	
Z	1	-4.5	-4	0	0	0	0
u	0	30	12	1	0	0	6000
v	0	10	8	0	1	0	2600
w	0	4	8	0	0	1	2000

Final Tableau

	Z	x	y	u	v	w	
Z	1	0	0	**0**	**5/12**	**1/12**	1250
x	0	1	0	**0**	**1/6**	**-1/6**	100
y	0	0	1	**0**	**-1/12**	**5/24**	200
u	0	0	0	**1**	**-4**	**5/2**	600

$$(0, \frac{5}{12}, \frac{1}{12}) \cdot (6000, 2600, 2000) = \frac{0+13000+2000}{12} = \frac{15000}{12} = 1250$$

$$\begin{pmatrix} 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} 6000 \\ 2600 \\ 2000 \end{pmatrix} = \begin{pmatrix} 100 \\ 200 \\ 600 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{5}{12} & \frac{1}{12} \\ 0 & 1/6 & -1/6 \\ 0 & -1/12 & 5/24 \\ 1 & -4 & 5/2 \end{pmatrix} \begin{pmatrix} C \\ S \\ B \end{pmatrix} = \begin{pmatrix} \frac{5}{12}S + \frac{1}{12}B \\ \frac{S-B}{6} \\ \frac{1}{12}S + \frac{5}{24}B \\ C - 4S + \frac{5}{2}B \end{pmatrix}$$

A Peek Into Sensitivity Analysis

Suppose initial amounts of cheeses are
C = 6000, S = 3000, B = 2760.

Then

$$\begin{pmatrix} 0 & \frac{5}{12} & \frac{1}{12} \\ \mathbf{0} & \mathbf{1/6} & \mathbf{-1/6} \\ \mathbf{0} & \mathbf{-1/12} & \mathbf{5/24} \\ \mathbf{1} & \mathbf{-4} & \mathbf{5/2} \end{pmatrix} \begin{pmatrix} 6000 \\ 3000 \\ 2760 \end{pmatrix} = \begin{pmatrix} 1480 \\ 40 \\ 325 \\ 900 \end{pmatrix}$$

Optimal solution is $x = 40, y = 325, u = 900$,Revenue = 1480.

$$(C, S, B) = \left(0, \frac{5}{12}, \frac{1}{12}\right) \text{ are}$$

Shadow Prices

They not only give us the optimal solution to the dual problem; they also tell us how much each additional ounce of cheese would boost the objective function in the primal problem.

Primal Problem	*Dual Problem*
Maximize $Z = cx$	Minimize $W = yb$
subject to	subject to
$Ax \leq b$	$yA \geq c$
and $x \geq 0$	and $y \geq 0$.

Note: If you like to write decision variables (unknowns) on the right, then you can write
$A^T y^T \geq c^T$ instead of $yA \geq c$

Primal Problem	*Dual Problem*
Maximize $Z = cx$	Minimize $W = yb$
subject to	subject to
$Ax \leq b$	$yA \geq c$
and $x \geq 0$.	and $y \geq 0$.
$\mathbf{x}$: column vector	$\mathbf{y}$: row vector

Alternative 1 for Dual	*Alternative 2 for Dual*
Minimize $W = \mathbf{b}^T\mathbf{w}$	Maximize $W = -\mathbf{b}^T\mathbf{w}$
subject to	subject to
$A^T\mathbf{w} \geq \mathbf{c}^T$	$-A^T\mathbf{w} \leq -\mathbf{c}^T$
and $\mathbf{w} \geq 0$	and $\mathbf{w} \geq 0$.
$\mathbf{w}$: column vector	$\mathbf{w}$: column vector

Theorem: The dual of the dual is the primal.

Dual of the Dual is the Primal

Primal	*Alternative 2 for Dual*
Maximize $Z = \mathbf{cx}$	Maximize $W = -\mathbf{b}^T\mathbf{w}$
subject to	subject to
$A\mathbf{x} \leq \mathbf{b}^T$	$-A^T\mathbf{w} \leq -\mathbf{c}^T$
and $\mathbf{x} \geq 0$	and $\mathbf{w} \geq 0$.
$\mathbf{x}$: column vector	$\mathbf{w}$: column vector