MATH 318: Operations Research
Spring 2023

Assignment 9
Due: Wednesday, April 26

READ: Hillier & Lieberman: Chapter 10, Section 5

PROBLEMS: Write up clear and complete solutions for the following problems from Hillier and Lieberman:

- 10.5-1 (Demonstrate that your answer is optimal by exhibiting the min cut.)
- **10.5-3ab**

I. Use Dijkstra's Shortest Path Algorithm to solve the Use Car Strategy Problem presented in class. Your solution should present a written explanation of what is the best "keep *vs.* trade-in" policies to follow if you wish to minimize your total costs over (a) a 1 year time period, (b) a 2 year time period, (c) a 3 year time period, (d) a 4 year time period, and (e) a 5 year time period.

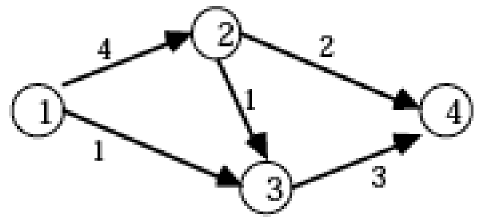
II. The Schmitt, Peterson, Schumer and Bremser families want to drive from Middlebury to New York to celebrate the end of the Spring Term together and catch some Broadway shows. Four cars are available. The Buick can carry four people, The Neon can carry three, the Camry can carry three, and the Mercury Sable Station Wagon can carry four. There are four people in each family who will make the trip. Since long car trips can be stressful on families, it has been decided that no car can carry more than two people from any one family.

(a) Formulate the problem of transporting the maximum number of people to New York as a maximum flow problem. *Hint*: represent each family as a node and each car as a node. Now add a source node and a sink node.

(b) Find the maximum number of people who can make the trip together. If you can find a solution by inspection then you do not have to go through the augmenting path algorithm step by step, but do indicate how you know that your solution is a maximum.

III.

Consider the network shown on the right.
Node 1 is the source, node 4 is the sink, and capacity numbers are given on the arcs.



(a) Find the maximum flow in the network. Identify min cut to prove optimality.

(b) Here is a linear programming formulation of the max flow problem:

(P) max $z = x_{12}+x_{13}$

such that

$$
\begin{array}{ccccccl}
x_{12} & & & & & \le 4 & \text{arc (1,2) capacity} \\
& x_{13} & & & & \le 1 & \text{arc (1,3) capacity} \\
& & x_{23} & & & \le 1 & \text{arc (2,3) capacity} \\
& & & x_{24} & & \le 2 & \text{arc (2,4) capacity} \\
& & & & x_{34} & \le 3 & \text{arc (3,4) capacity} \\
x_{12} & & -x_{23} & -x_{24} & & = 0 & \text{node 2 flow balance} \\
& x_{13} & +x_{23} & & -x_{34} & = 0 & \text{node 3 flow balance}
\end{array}
$$

all $x_{ij} \geq 0$.

The objective function represents the flow out of the source. Show that the constraints algebraically imply that it is equal to the flow into the sink.

(c) Form the dual of (P). Let u_i be the dual variable associated with the node i constraint, and w_{ij} be the dual variable associated with the arc (i, j) constraint.

(d) For your cut K from (a), verify that setting $u_i = \begin{cases} 1, i \in S \\ 0, i \notin S \end{cases}$ and $w_{ij} = \begin{cases} 1, (i,j) \in K \\ 0, (i,j) \notin K \end{cases}$

gives a dual feasible solution (that is, just follow this recipe and verify that it satisfies the dual constraints). Here S is the set of nodes which are the initial nodes of arcs in the cut K; that is, i is in S means the directed arc (i,j) is in K for some node j. What is the interpretation of its objective value?